Importance measures for the dynamic flowgraph methodology

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Summary

This report surveys some existing risk importance and sensitivity measures with the view of finding an importance measure suitable for use with the dynamic flowgraph methodology (DFM).

First, requirements for importance measures in the context of DFM are considered. DFM models are dynamic, discrete and a DFM variable may assume its state from more than two alternatives. They are analysed against a top event that is a conjunction of DFM variable values at different time instances.

A literature survey is made of importance measures for various reliability models. Importance measures equivalent or analogical to Fussell-Vesely, Birnbaum, risk reduction worth, risk achievement worth and criticality importance are formulated in the DFM setting. The developed measures are illustrated by a simple example taken from the literature, describing the failure of a space-based nuclear reactor control system. Defining system sensitivity in a meaningful way in DFM models, applying importance measures developed for Markov models in a DFM setting, and defining importance measures for DFM multi-phase models are open research questions.

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1 Introduction

The purpose of importance measures (IM) is to assess how much a single component, a subsystem, a basic event, a part of a process or some such contributes to the failure risk of a system. This contribution may be analysed for a specific time instance (time-dependent importance measures), over the mission time of the system (time-independent importance measures), or disregarding probabilities (structural importance measures).

A closely associated concept is that of sensitivity measures. They purport to measure how much the failure risk of the whole system changes when small changes are made to the failure probability of the component under study.

Importance measures can be put to use in several ways. In risk management, IMs indicate subsystems where reliability improvement will pay high dividends in terms of lowered risk for the whole system; thus the risk profile of a system can be balanced. They are also used in safety classification. They can be used in the optimization of test intervals and allowed repair times. If risk modeling is still underway (or perhaps has been taken into reconsideration), IMs tell which parts of the system to model more accurately, and which parts can be treated more cursorily.

Thus, for any probabilistic risk assessment method, a reliability measure that fits the method will significantly increase its applicability. It is therefore of great theoretical and practical interest to find an IM suitable for the dynamic flowgraph methodology (DFM), because none have been proposed so far in the literature.

The purpose of this short notice is twofold. First, an attempt is made to formulate some requirements for an IM and properties of a good one, all regarding the DFM. Second, a small literature survey is made to examine to what degree existing IMs, developed for other models, would suit DFM.

It is assumed that the reader knows the basics of both DFM and IMs in reliability theory. For DFM, some introductions are found in [16] and [19]. For IMs, the basics can be found in [4] and [26].

2 Requirements of importance measures for DFM

2.1 General requirements

The practical use of IM's induces some general requirements. Some of these are

- Robustness. When comparing e.g. component IM's, it is usually required that the ranking provided by the IM is insensitive to uncertainties. This robustness requirement is usually better filled by relative IM's than absolute IM's.

- Informativeness. An importance measure should have meaning and be descriptive to the user and within the decision problem at hand. For example, IM's that measure how total risk changes when a component becomes more or less reliable are informative. On the other hand, IM's that assume all failure probabilities to be equal are not.

- Scope, versatility and adaptability. For some decision problems, only a few importance measures are available; for some others, the spectrum of IM’s is much larger. An importance measure that is applicable to a variety of tasks is usually preferred by the user to one that isn’t. Examples of this applicability are use for measuring the importance of groups of components instead of a single component, use for both fault trees and for reliability block diagrams, use for both static and dynamic reliability models, etc. The fewer assumptions an IM requires, the more adaptable it is. It is a merit to an IM that it can easily be adapted to new situations.
- Ease of computation. Methods that don’t set high requirements on computational resources are better than ones that do.

2.2 Special features of DFM models

DFM models are dynamic, i.e., the values of variables at a given time instance may depend on the values of some variables at previous time instances. Thus, importance measures have to take into account the time dimension in one way or another. In this context, the time dimension doesn't include the whole system's mission time but just the time span of the events leading to a top event. This time span naturally varies with the top event.

Another facet of DFM models' dynamicity is that they may contain feedback loops. Thus, they may in principle contain arbitrarily lengthy sequences leading to a failure. Further, choosing the number of time steps is part of DFM modelling, but currently there exists no formal method to find out the shortest time span that produces all relevant prime implicants for a top event. A method for recognizing prime implicants that reoccur when the number of time steps is increased, and a method for recognizing patterns of prime implicants and when all relevant patterns have been found (i.e. increasing the number of time steps doesn’t produce new patterns) are needed. The former task is easy, the latter less so.

The variables of DFM models may assume their value from sets of more than two possible values. None of these values may necessarily be interpreted as failure; rather, a failure may consist e.g. of a variable staying in a given value for too many consecutive time instances. Thus, the works/fails interpretation generally applicable to traditional reliability models cannot generally be applied to DFM (although it is often possible to formulate a DFM model in such a way that some states of some variables can be interpreted as failure). In particular, the concept of failure probability of a component doesn't necessarily have a meaning for all variables in DFM.

A DFM model is a finite-state machine representation of the system. Its variables don’t necessarily represent all the relevant features of a component. Thus, DFM's use of variables – rather than components – is a diversion from classical reliability. A single component (e.g. a pump) might be described by several variables (e.g pressure, flow, electrical current intake etc.). Therefore, when a component's reliability is an issue, it might not be straightforward to figure out how traditional IMs should be applied; again, group importance measures might help here.

2.3 DFM and other reliability models

There are three kinds of traditional system failure probability models for which importance measures have been proposed: reliability block diagrams, fault trees and Markov models.

Reliability block diagrams – for which structural IMs are often determined – are structured as directed graphs, and so are DFM models. However, there are important differences, too. Reliability block diagrams have an input and an output, and the system works if there is a path from the input to the output. DFM models don't necessarily have a well-defined input or output, and there are no well-defined works/fails states; rather, the user of a DFM model defines a condition of interest as a conjunction of (variable, value, time instance) triplets as a top event, and DFM computation consists of finding prime implicants leading to the top event. However, the top event need not describe a failure of the system; rather, it is any condition of interest.

Fault trees differ from DFM in three essential ways. First, they are static. Second, their variables are Boolean. Third, each fault tree corresponds to a top event, and thus top events and system models are usually inseparable in them (a fault tree can be constructed modularly to enable reuse, but this modularity isn't an inherent property of fault trees). These differences mean that many central concepts of fault trees – such as coherence and relevance – don't apply to DFM's. The relevance of individual variables in DFM varies, because the participation of individual variables in DFM to the prime implicants varies with the top event; one might define relevance of a variable relative to a top event, though.

Markov models are closest to DFM models of the three classical system failure probability models. Thus, it would seem that IMs developed for Markov models would be especially good candidates for DFM IMs.

3 A brief look at the literature

Importance measures have been the subject of active research interest since late 1960's. The purpose of this overview is to find IMs that would have meaning and suitability for application to DFM, and therefore only a small selection of research concerning IMs is surveyed.

We go through the IMs that were deemed most significant by experts in [29]; these can be assumed to be the most used IMs in practice . These include the risk reduction worth (RRW), risk achievement worth (RAW), Fussell-Vesely, Birnbaum, criticality importance, and partial derivative (sensitivity) measures. Further, we take a look at IMs for multi-state systems and Markov models, since these model classes bear some similarity to DFM. A promising IM introduced since the writing of [29], the differential importance measure (DIM), will also be discussed briefly. Also time-independent IMs will be discussed.

In what follows, a basic event is understood to mean that a component fails.

The set of prime implicants (or minimal cut sets if the system is coherent) contains all relevant information about the reliability of a system with respect to a top event. Therefore it is sufficient to consider IMs that apply to the set of prime implicants of a system, and this applies to both structural and probabilistic IMs.

However, when one seeks IMs that are independent of a top event, observing prime implicants alone is not sufficient.

A prime implicant in DFM is a logical conjunction of formulae $\varphi_{i.}$ Each of these formulae indicates the logical statement that a certain time-dependent variable *x(t)* has a certain value *v* at a certain time instance *T*: $\phi_i : \left(x_j \left(T_k \right) = v_{l,j} \right)$ for some *j*, *k* and *l*. For example, the fact that the value of the variable "valve position" is "open" at time *T*=-1 might be expressed as the formula ValvePosition(-1)=Open. The value v_l that $x_i(t)$ takes, satisfies $v_l \in \Upsilon_i$; here Υ_i is the set of possible values for $x_i(t)$. For example, the set of the valve position's possible values could be $\Upsilon_j = \{\text{open, semiopen, closed}\}$.The formulae of a prime implicant are unique in the sense that no two formulae within a prime implicant share the same *j* and *k*; that is, a variable cannot have two distinct values at the same time instance within a prime implicant. A single prime implicant is then of the form

$$PI_k = \varphi_{k_1} \wedge \varphi_{k_2} \wedge \ldots \wedge \varphi_{k_{m_k}} \tag{1}$$

Where m_k is the cardinality of the k^{th} prime implicant (the number of formulae in it). The set of prime implicants for a DFM model and a top event can then be listed as

$$PI = \{PI_1, PI_2, \ldots, PI_n\} \tag{2}$$

And the probability of the top event occurring is

$$P(TOP) = P(PI_1 \vee PI_2 \vee \ldots \vee PI_n) \tag{3}$$

If the probabilities

$$P(x_i(T) = v_j) \overset{def}{=} p_{i,j,T} \tag{4}$$

are known and their variables independent, the probability of the prime implicant PI_k happening is simply the (arithmetic) product of the probabilities of its indicator variables, $p_{i,j,t}$. That is,

$$P(PI_k) = p_{i_1,j_1,T_1} \cdot p_{i_2,j_2,T_2} \cdots p_{i_{m_k},j_{m_k},T_{m_k}} \tag{5}$$

The probability of PI happening – i.e. that any of the prime implicants in it happens - can be computed in the usual manner using the inclusion-exclusion development, the method of disjoint products or Kitt's method (see [21], chapter 5).

There are two kinds of IMs: structural and probabilistic. Structural importance factors measure the importance of a component from the system's topology (expressed by reliability block diagram, flow diagram etc.) and other structural information, without reference to actual probabilities. Probabilistic IMs describe the amount that a single component (or a group of components) contributes to the total failure probability of a system.

3.1 Structural importance measures

The main benefit of structural importance measures is that no information about actual probabilities is needed. Here we consider the well-known Birnbaum's measure of structural importance and the structural importance measure of Barlow and Proschan. Some other structural importance measures are cut-importance and node criticality, and a lesser-known but promising measure called joint criticality. An overview of these can be found in [22].

3.1.1 Birnbaum's measure of structural importance

Birnbaum's measure of structural importance [6], $I_{B,\phi}$, is the oldest and perhaps the best-known structural IM. It is the proportion of state vectors where the component x_i is critical – i.e. where changing x_i from working to non-working would result to the system failure – to all state vectors where x_i works (of which there are 2^{n-1} in an n-component system).

$I_{B,\phi}$ cannot be directly applied to DFM models because there isn't necessarily any state of a variable that can be interpreted as failure (if there were, $I_{B,\phi}$ could be applied as such). However, an analogous measure can be defined as

$$I_{B,DFM}(x_i) = \frac{Card(\{PI_k | x_i \in PI_k, PI_k \in PI\})}{Card(PI)} \quad (6)$$

where the numerator is the number of prime implicants containing the variable c_i (note that in the context of DFM we talk about variables, not components), and the denominator is the total number of prime implicants (PI) of the top event. Here we denote $x_i \in PI_k$ whenever PI_k contains a condition of the form $x_i(t) = v$ for some t and v.

$I_{B,DFM}$ can easily be extended to a group importance measure: if several DFM variables correspond to the component, then the numerator is the number of prime implicants containing any of these variables.

3.1.2 Structural importance measure of Barlow and Proschan

The structural importance measure of Barlow and Proschan [3] is based on the assumption that all components have a common reliability p_0, and that p_0 is distributed uniformly over [0,1]. It can be defined as

$$I_{BP}(x_i) = \int_0^1 \left(R(1_i, \mathbf{p}^i) - R(0_i, \mathbf{p}^i)\right) dp_0 \quad (7)$$

Where $R(1_i, \mathbf{p}^i)$ is the system failure probability when $x_i = 1$ and $\mathbf{p}^i$ is the vector of probabilities $p_j = P(x_j = 1)$, with the assumption that $p_j = p_0$ for $j \neq i$; $R(0_i, \mathbf{p}^i)$ is defined correspondingly.
A similar importance measure can easily be formulated for DFM, if we assume that we consider the importance of a DFM variable within a formula only. Let this

formula be $\varphi_j \equiv (x_i(T_k) = v_l(x_i))$. Further, let us denote with $R(P(\varphi_j) = 1)$ the system failure probability when φ_j surely happens, and with $R(P(\varphi_j) = 0)$ the system failure probability when φ_j surely doesn't happen. Then a modified version of the Barlow-Proschan measure for DFM can be defined as

$$I_{BP,DFM}(\varphi_i) = \int_0^1 \left(R(P(\varphi_j) = 1) - R(P(\varphi_j) = 0) \right) dp_0 \tag{8}$$

In the more general setting, when we are considering the importance of a variable in general, or a group of variables, the situation becomes more complicated. It would be straightforward to set $p_{i,j,t}$ =1 for all j and t in $R(1_i, \mathbf{p}^i)$, and $p_{i,j,t}$ =0 for all j and t in $R(0_i, \mathbf{p}^i)$. However, the same variable can take different values at the same time instance. Setting the probability of all these events would be contradictory, because they clearly are mutually exclusive.
A way out of this quandary is to consider *R* when at least one of the formulae containing x_i holds versus *R* when none of them hold. Let us denote the set of formulae in *PI* that have x_i as their variable by
$\Phi(x_i) = \{\varphi_j : \exists t, v \text{ such that } \varphi_j \equiv (x_i(t) = v)\}$. Then we can define

$$I_{BP,DFM}(x_i) = \int_0^1 \left(R(\vee \varphi_j : \varphi_j \in \Phi(x_i)) - R(\wedge \neg \varphi_j : \varphi_j \in \Phi(x_i)) \right) dp_0 \tag{9}$$

Computing this integral is conceptually easy, because $R(\vee \phi_j : \phi_j \in \Phi(x_i))$ and $R(\wedge \neg \phi_j : \phi_j \in \Phi(x_i))$ are polynomials of the probabilities pi,j,t. when the variables are independent. However, computations may become quite involved because when calculating the system failure probability with the conjunction $\vee \varphi_j$ in effect, we may have quite many alternatives to go through.
A somewhat lighter alternative is to consider the maximal and minimal impacts that the variable can make, given that the other variables are integrated as in (8). For illustration purposes, let us assume that the variable x_i can be in two states, and that the probabilities $p_{i,1}$ and $p_{i,2}$ of these states appear in the prime implicants. $p_{i,1}$ and $p_{i,2}$ are interrelated in the sense that $p_{i,1} + p_{i,2} \leq 1$. Then we can define

$$I_{BP,DFM}(x_i) = \int_0^1 \cdots \int_0^{1-p_{i,1}} \cdots \int_0^1 \left(\max_{v_i \in \Upsilon_i} R(x_i) - \min_{v_i \in \Upsilon_i} R(x_i) \right) dp_0 \cdots dp_{i,2} \cdots dp_n \tag{10}$$

It is easy to generalize this to the case where a variable may be in several states, which appear in the prime implicants. Then the constraint set by mutual exclusivity is $\sum_j p_{i,j} \leq 1$, and the upper bound in the integrals are of the form

$$I_{BP,DFM}(x_i) = \max_{v_i \in \Upsilon_i} \int_0^1 \cdots \int_0^{1-p_{k,1}} \int_0^{1-p_{k,1}-p_{k,2}} \cdots \int_0^1 R(x_i) dp_0 \cdots dp_{k,3} dp_{k,2} \cdots dp_n \\ - \min_{v_i \in \Upsilon_i} \int_0^1 \cdots \int_0^{1-p_{k,1}} \int_0^{1-p_{k,1}-p_{k,2}} \cdots \int_0^1 R(x_i) dp_0 \cdots dp_{k,3} dp_{k,2} \cdots dp_n \tag{11}$$

An example of the calculation is given in section 4.

3.2 Importance measures for fault trees

Importance measures for fault trees are central to the subject of this report, but well-known. The interested reader can find more on these in, e.g., [6][11][12][29].

3.2.1 Fussell-Vesely

Perhaps the most well-known importance measure for fault trees is the Fussell-Vesely importance measure (FV). It can be defined as the conditional probability that a minimal cut set containing the component c_i has failed, given that the system has failed [26]. Utilizing the well-known formula for conditional probability, it is easy to see that

$$FV_i(\mathbf{q}) = \frac{R_i}{R} = \frac{Fr\left(\bigcup_{j=1}^{m} MCS_j^i\right)}{Fr\left(\bigcup_{k=1}^{n} MCS_k\right)} \tag{12}$$

Here $\mathbf{q}$ is the vector of component failure probabilities, R_i is the probability that a cut set containing the component c_i fails (here expressed as the frequency of minimal cut sets containing c_i), and R is the system failure probability (here expressed as the frequency of minimal cut sets). The FV depends on the top event because it is a function of minimal cut sets.

Formulation of FV for DFM is straightforward. Just replace the minimal cut sets by prime implicants in (12), noticing that a prime implicant is listed in the numerator if it contains the variable with any value at any time instance. The probabilities (frequencies) in the numerator and denominator can easily be computed (see the beginning of this chapter, page 5).

$$DFMFV_i(\mathbf{q}) = \frac{R_i}{R} = \frac{P\left(\bigcup_{j=1}^{m} \left\{PI_j \middle| x_i \in PI_j\right\}\right)}{P\left(\bigcup_{k=1}^{n} PI_k\right)} = \frac{P\left(\bigcup_{j=1}^{m} PI_i^j\right)}{P(PI)} \tag{13}$$

Here PI_i^j denotes a prime implicant of the top event that contains the variable x_i.

It is easy to extend FV to calculating the importance of a component in situations where several DFM variables correspond to a single component (e.g. the pressure and flow in a pipe). The only thing to be changed is that all prime implicants containing those variables are included in the numerator in (12). This can be seen as the Fussell-Vesely group importance measure for DFM.

3.2.2 Birnbaum

The Birnbaum importance measure [6] can be defined as

$$IB_i = R(x_i = 1) - R(x_i = 0) \tag{14}$$

Here, R(x_i=1) is the probability that the system fails given that the component c_i has failed, and R(x_i=0) is the probability that the system fails given that the component c_i is assumed perfectly reliable (i.e. the corresponding basic event never to happen).

It is evident that we cannot define a similar IM for DFM because DFM variables don't necessarily assume works/fails states (of course, in a DFM model where this interpretation can be made, the ordinary Birnbaum importance measure can be used). However, an analogous IM can be defined using the set of prime implicants for a top event. The starting point of the generalized measure is that the Birnbaum measure represents the maximal and minimal values that the top event probability can take, when the state of the variable x_i can be chosen freely.

Let $P\left(x_i(T_k) = v_{i,j}\right) = 1$ mean that variable x_i surely yields the value $v_{i,j}$ at the time instance T_k. Similarly, let $P\left(x_i(T_k) = v_{i,j}\right) = 0$ mean that x_i surely doesn't yield the value $v_{i,j}$ at the time instance T_k. Then, assuming e.g. that variable $\varphi_2 \equiv x_i(T_k) = v_{i,j}$ and $P\left(x_i(T_k) = v_{i,j}\right) = 1$ in (5),

$$P\left(PI_k \middle| x_i(T_k) = v_{i,j}\right) = P\left(\phi_1\right) \cdot 1 \cdots P\left(\phi_{m_k}\right) \tag{15}$$

and

$$P\left(PI_k \middle| x_i(T_k) = v_{i,j}\right) = P\left(\phi_1\right) \cdot 0 \cdots P\left(\phi_{m_k}\right) = 0 \tag{16}$$

Further, denote by $v_j\left(x_i(T_k)\right)$ the event that x_i yields the value $v_{i,j}$ at time instance T_k. Finally, let $\left\{v_j\left(x_i(T_k)\right)\right\}$ denote a set of value assignments for x_i for all time instances T_k. Now we can define the Birnbaum importance measure for DFM as

$$DFMIB_i = \max_{\left\{v_j\left(x_i(T_k)\right)\right\}} R - \min_{\left\{v_j\left(x_i(T_k)\right)\right\}} R \tag{17}$$

That is, the Birnbaum measure for DFM is the risk of the top event occurring when the variable x_i is assigned the worst possible value minus the risk of the top event occurring when the variable x_i is assigned the best possible value at each time instance (note that the worst and best values can be different at different time instances).

Extending this Birnbaum measure for DFM to groups of variables is straightforward. If the group contains e.g. the variables x_i and x_l, the group Birnbaum measure for DFM is

$$GroupDFMIB_{i,l} = \max_{\{v_j(x_i(T_k))\}\times\{v_j(x_l(T_k))\}} R - \min_{\{v_j(x_i(T_k))\}\times\{v_j(x_l(T_k))\}} R \tag{18}$$

A remark on the computational effort of these maximization and minimization tasks is in order. If we assume that the variable has n possible states, and the DFM model is considered for m steps, there are only mn evaluations to be made for a complete evaluation of all the possibilities. Typically, both m and n are small, and therefore the maximization and minimization tasks are computationally easy.

3.2.3 Risk reduction worth

Risk reduction means the reduction in system failure probability when the component (or basic event) we are analyzing surely works (or surely doesn't occur):

$$RR_i = R - R(x_i = 0) \tag{19}$$

Here, as above, R is the nominal system failure probability, and $R(x_i=0)$ is the probability that the system fails given that the basic event x_i surely doesn't occur (or the component is perfectly reliable).

Risk reduction worth is the ratio of these failure probabilities, i.e.

$$RRW_i = \frac{R}{R(x_i = 0)} \tag{20}$$

Using the ideas of section 3.2.2, it is easy to formulate similar IMs for DFM. For risk reduction this becomes

$$DFMRR_i = R - R(\varphi(x_i) = 0) \tag{21}$$

That is, we consider the difference between the nominal system failure probability and the failure probability when variable i is assumed to receive values that minimize risk.

For risk reduction worth this becomes

$$RRW_i = \frac{R}{R(\varphi(x_i) = 0)} \tag{22}$$

Again as in section 3.2.2, it is straightforward to extend these IMs to group importance measures.

3.2.4 Risk achievement worth

Risk achievement means the difference between the system failure probability when the component/variable/basic event *i* is perfectly reliable or never occurs, and the nominal system failure probability.

$$RA_i = R(x_i = 1) - R \tag{23}$$

Analogously to section 0, risk achievement worth is the ratio of the two risks:

$$RAW_i = \frac{R(x_i = 1)}{R} \tag{24}$$

Extending these to DFM proceeds in the manner of section 0.

3.2.5 Criticality importance

Criticality importance [14] is an extension of the Birnbaum importance measure. It takes into account the risk that component *i* fails (or variable x_i=1).

$$CR_i = \frac{R(x_i = 1) - R(x_i = 0)}{R} P(x_i = 1) = \frac{IB_i}{R} p_i \tag{25}$$

Where p_i is the failure probability of component *i*.

When trying to find an analogue of criticality importance for DFM models, the first problem is that there isn't necessarily any state of any variable in a DFM model that can be interpreted as failure state. However, if we just want to analyse a certain state of a certain variable (regardless of whether it is a failure state or not), and if the failure probability of the component remains constant over the time span considered, it is again straightforward to formulate a DFM version of this IM (see section 3.2.2).

On the other hand, if the failure probability of the component varies with time during the timespan considered in the DFM model, things get more complicated. Which probability of the $p_i(t)$ at different time instances should we take to the IM? The mean of $p_i(t)$ in the time period considered? The maximum of $p_i(t)$ in the time period? This is an interesting research question, but of rather little practical relevance, and therefore it will not be pursued further here.

3.2.6 Sensitivity

Sensitivity is the partial derivative of the system's failure probability with respect to the failure probability of component *i*:

$$PD_i = \frac{\partial R}{\partial p_i} \tag{26}$$

When applying this to DFM, the first question concerns the time extension: should we apply to the variable at a single time instance or at all time instances? If it is applied at a single time instance, the partial derivative can be calculated easily at least in principle: just find the total failure probability of the system in terms of the prime implicants – a sum of multinomials of the p_i's – take only the terms containing p_i, factorize p_i out of the multinomials and compute the value of the resulting expression. Usually one is more interested in the variable's overall contribution to the total failure probability. However, the variable's values at different time instances might make a different contribution to the total failure probatility. Therefore, equating the variable's values at different time instances doesn't seem like a good idea.

If the variable receives a given value at all the time points considered with equal probability, this problem is solved. Again, one can use the total risk as a multinomial of the individual probabilities, and calculate its derivative with respect to the probability.

The second question concerns the range of the variable. Should we consider the variable at different states as equal? Again, the variable at different states make a different contribution to the failure probability, but the sensitivity of each state can be calculated from the set of prime implicants. This sensitivity corresponds to type 2 importance measures in multistate system reliability analysis (section 3.3).

One way around these problems might be to adopt the approach taken in the calculus of variations: find the maximal contribution that small perturbations to the variable can make. However, since DFM variables are discrete-valued (and the values of a variable are unordered), this approach is problematic.

Another way is not to try find the sensitivity of system failure probability to a variable, but instead to a parameter – perhaps shared by several variables. Such a parameter might be human error probability, maintenance interval etc. Again, finding sensitivity in this case is straightforward.

All in all, calculating the sensitivity of total risk to variations in individual variables is a difficult research question. However, when the failure probabilities don't vary with time, and we settle to finding the sensitivity of the total risk to the variable at a given value, the problem is easily solvable.

3.2.7 Differential importance measure

The differential importance measure (DIM) [9] is a relatively new measure with certain attractive properties, such as additivity (i.e. The DIM of a group of basic events or parameters is the sum of individual DIMs). Its purpose is to measure the importance of proposed changes that affect component properties and multiple basic events. It is the fraction of the total change in the system's failure probability that is due to a change in the parameter (or basic event) considered.

$$DIM(x_i) = \frac{\frac{\partial R}{\partial x_i} dx_i}{\sum_{j=1}^{n} \frac{\partial R}{\partial x_j} dx_j} \tag{27}$$

Again, we run into the same problems in the general DFM setting as in calculating sensitivity (section 3.2.6): it is not clear how to treat a variable in different states or at different time instances. If we again adopt the stance that DIM is computed for only a variable at a time instance, or for a parameter (such as maintenance period), it is quite straightforward to see how to calculate DIM for a DFM model.

3.3 Importance measures for multi-state systems

Multi-state systems in reliability theory are systems where components and the system aren't simply working or failed, but there are intermediate stages between these two extremes. There are two kinds of multi-state systems: those where the components have a binary state space (works/fails) but the system as a whole may be in one of several states; and those where the components, too, may be in more than two states.

If the system's state vector is **x** – each x_i corresponding to the state of component *i* – and $\phi(\mathbf{x})$ its structure function, mapping the states of the individual components to the total system's state, we may present the system's failure probability as

$$MR_d = P(\phi(\mathbf{x}) \geq d) \quad (28)$$

Where *d* is a constant demand that the system must satisfy. More general formulations of reliability for multi-state systems are also possible [12][25].

Note that a multi-state system can be converted to a Boolean system in a way that a decimal number can be converted to a binary number. It is also easy to convert the multi-state system's failure risk to a binary variable: the condition $\phi(\mathbf{x}) \geq d$ is a binary condition (either true or false). However, dealing with a multi-state system as such (rather than its Boolean equivalent) leads to better representational economy and conceptual clarity.

There are two kinds of importance measures for multistate systems [24]: type 1 IM's measure how a specific component affects the system's reliability. Birnbaum, RAW and FV can be directly extended to multistate setting [24]; however, these extensions only consider the possible state levels and not the probability of a component being in that state. [24] also developed a set of composite IM's that take the probabilities into account: mean absolute deviation, mean multi-state reliability achievement worth and mean multi-state Fussell-Vesely. [17] extended Birnbaum's measure to the multistate case via an importance vector, and [31] extended this to a componentwise IM.

Type 2 IM's measure how a particular state or set of states of a specific component affect multistate reliability. [32] extends RAW, RRW, Fussell-Vesely and Birnbaum to a multi-state setting using Monte Carlo simulation.

It is for the time being unclear whether, and to what extent, importance measures developed for multistate systems can be adapted for DFM. A significant conceptual difference between multistate systems and DFM models is that the latter don't consider performance levels but, rather, top events. If the DFM model represents a multistate system, the top event may be formulated in a way that describes failure to reach the required performance level.

3.4 Importance measures for Markov chains

Markov models are considered to be the main alternative to DFM in dynamic reliability analysis [1]. Therefore it is tempting to think that importance and sensitivity measures developed for Markov models would be applicable to DFM, too. However, the present approaches to the problem [27][28] seem to rely on perturbation of the transition rates matrix. Thus the application of Markov model importance measures to DFM boils down to finding equivalent of the transition rates matrix in DFM. Formulation of a meaningful transition rates matrix for a DFM model is an open research problem.

3.5 Time-dependent importance measures

Time-independent importance measures [5][23] aim to evaluate the importance of a component or a failure mode over the mission time of the system. The motivation for this is that the reliability of individual components varies with time, and thus, to get a comprehensive understanding of a component's contribution to the system's reliability we need to take into account the system's entire mission time.

The usual way to formulate a time-independent IM is to take a time-dependent IM and integrate it over the system's mission time. This is possible for a DFM model, too, if the nature of the individual probabilities' time-dependence of the model is known.

4 Application example

Garrett and Apostolakis present a case study of applying DFM to a space-based reactor control system [15]. The reactor provides power to a spacecraft operating in a strategic defense initiative (SDI) environment. The control system controls the power production of the reactor by moving its control rods if measurements from the reactor imply a need for it. The purpose of the control system is to start or stop the system at the operator’s command, to maintain the desired power level, and to ensure that the fuel temperature does not exceed a given maximum value. The control system raises the power level by raising the control rods, lowers it by lowering the control rods, and monitors the reactor attributes (fuel temperature, power level and startup rate). A human operator specifies the target power level, and may issue start and stop commands.

The central variables of the DFM model are listed in Table 1.

variable	explanation	Relevant states
DR	State of the rod motion	in max – maximum movement speed
RP	Rod position	full in – rods are fully inserted
T	Fuel temperature	melt (>1800 C) – above the melting point of the fuel hot (1400, 1800 C) – above the shutdown override action temperature but below melting point
TS	Temperature sensor status	low – temperature sensor claims that fuel temperature is low null – temperature sensor doesn’t give a reading
MS	Rod drive motor status	stalled

Table 1. variables of the example model

We assume that each of these variables can yield some other states. For example, the fuel temperature can yield a value smaller than 1400 degrees. For all variables, we call these other states normal.

For illustration purposes, we use the following probabilities for each event that occurs in a prime implicant. The probability figures are invented. The abscissae of the figures have been chosen to be different prime numbers, so that the contribution of each to the top event can be more easily analyzed.

Variable	state	formula	probability	probability value
RP	Full in	φ_1	p_1	11E-5
T	hot	φ_2	p_2	7,00E-03
T	melt	φ_3	p_3	1,30E-04
TS	low	φ_4	p_4	3,00E-03
TS	null	φ_5	p_5	5,00E-05
MS	stalled	φ_6	p_6	1,70E-04

Table 2. Probabilities of each variable being in a certain state at time instance -1.

For brevity, we denote

$$P(\varphi_i) \overset{def}{=} p_i \quad (29)$$

Garrett and Apostolakis consider the following top event, corresponding to the reactor failing to scram upon high temperature:

$$\neg(DR(0) = \text{in max}) \wedge ((T(-1) = \text{hot}) \vee (T(-1) = \text{melt})) \quad (30)$$

For this top event, they obtain the seven prime implicants listed in Table 3.

PI #	PI	Expressed as events
1	(*RP*(-1)=full in) $\wedge$ (*T*(-1)=hot)	$\varphi_1 \wedge \varphi_2$
2	(*RP*(-1)=full in) $\wedge$ (*T*(-1)=melt)	$\varphi_1 \wedge \varphi_3$
3	(*TS*(-1)=low) $\wedge$ (*T*(-1)=hot)	$\varphi_2 \wedge \varphi_4$
4	(*TS*(-1)=null) $\wedge$ (*T*(-1)=hot)	$\varphi_2 \wedge \varphi_5$
5	(*TS*(-1)=null) $\wedge$ (*T*(-1)=melt)	$\varphi_3 \wedge \varphi_5$
6	(*MS*(-1)=stalled) $\wedge$ (*T*(-1)=hot)	$\varphi_2 \wedge \varphi_6$
7	(*MS*(-1)=stalled) $\wedge$ (*T*(-1)=melt)	$\varphi_3 \wedge \varphi_6$

Table 3. The prime implicants of the top event (30)

First we calculate an expression for the total risk R using the inclusion-exclusion development. Remember that $R = P(TOP) = P(PI_1 \vee \ldots \vee PI_7)$. In principle, we should apply the inclusion-exclusion development to all PI's subsets, the number of which is 2^7=128. However, we note for example that PI_1 and PI_2 cannot occur simultaneously because the formulae (*T*(-1)=hot) and (*T*(-1)=melt) are mutually exclusive. Taking all such exclusivity constraints into account, we obtain

$$\sum_{i=1}^{7} P(PI_i)$$

$$R = \begin{array}{l} -(P(PI_1 \wedge PI_3) + P(PI_1 \wedge PI_4) + P(PI_1 \wedge PI_6) + P(PI_2 \wedge PI_5) \\ +P(PI_2 \wedge PI_7) + P(PI_3 \wedge PI_6) + P(PI_4 \wedge PI_6) + P(PI_5 \wedge PI_7)) \\ +P(PI_1 \wedge PI_3 \wedge PI_6) + P(PI_1 \wedge PI_4 \wedge PI_6) + P(PI_2 \wedge PI_5 \wedge PI_7) \end{array} \quad (31)$$

Assuming independence between the formulae we can calculate the individual probabilities in (31). For example,

$$P(PI_1) = p_1 p_2 \quad (32)$$

Then, the risk R can be expressed in terms of formulae as

$$R = \begin{array}{c} p_1p_2 + p_1p_3 + p_2p_4 + p_2p_5 + p_3p_5 + p_2p_6 + p_3p_6 - (p_1p_2p_4 + \\ p_1p_2p_5 + p_1p_2p_6 + p_1p_3p_5 + p_1p_3p_6 + p_2p_4p_6 + p_2p_5p_6 + p_3p_5p_6) \\ +p_1p_2p_4p_6 + p_1p_2p_5p_6 + p_1p_3p_5p_6 \end{array} \quad (33)$$

R can be computed from (33) and Table 2 to be 2,33E-05.
Now we can calculate the values of structural importance measures for each of these variables.

For the Birnbaum measure of structural importance, this is easy. Just calculate the number of prime implicants where probabilities corresponding to the variable exist, and divide that by the total number of prime implicants.
For the Barlow-Proschan measure, the calculation is somewhat more involved. For illustration purposes, let us consider the calculation of the measure for the variable *TS*. The set of possible states $\Upsilon_{TS} = \{\text{normal, low, null}\}$. The only other variable that takes two distinct values in the prime implicants is T.
We get the following integral that has to be evaluated for each of the possible states of *TS*:

$$I_{BP,DFM}(TS) = \max_{TS \in \{normal,low,null\}} \int_0^1 \int_0^1 \int_0^{1-p_2} \int_0^1 (R(TS)) dp_6 dp_3 dp_2 dp_1 - \min_{TS \in \{normal,low,null\}} \int_0^1 \int_0^1 \int_0^{1-p_2} \int_0^1 (R(TS)) dp_6 dp_3 dp_2 dp_1 \quad (34)$$

We get the following possibilities for the integral, from which we select the smallest and the largest:

TS	$P(\varphi_4)$	$P(\varphi_5)$	$\int_0^1 \int_0^1 \int_0^{1-p_2} \int_0^1 (R(TS)) dp_6 dp_3 dp_2 dp_1$
low	1	0	7/24
null	0	1	1/3

normal	0	0	1/4

Table 4. Computation of the Barlow-Proschan structural importance measure for the variable TS

As readily seen from Table 4, the maximum is 1/3 and the minimum is 1/4, and so the Barlow-Proschan measure for *TS* is 1/12. A side result is that the maximum is obtained when *TS*=null.

The computation of the Barlow-Proschan importance measure for the other variables proceeds similarly. All in all, we get the following table.

Variable	Birnbaum	Barlow-Proschan
RP	2/7 = 0.286	1/24
T	7/7 = 1	11/24
TS	3/7 = 0.429	1/12
MS	2/7 = 0.286	1/24

Table 5. Some structural importance measures for the reactor control system model and top event (30)

As we see, *T* is the most important variable by both structural importance measures. This is natural since T occurs in every prime implicant. The two structural importance measures are consistent in the sense that they give the same ranking for the variables: T is the most important, TS the second, and RP and MS are third, being equally important. This naturally raises the hypothesis that the DFM analogues of Birnbaum and Barlow-Proschan structural importance measures defined in sections 3.1.1 and 3.1.2 are as consistent in more general settings.

Probabilistic importance measures can be computed similarly. Here we consider only the importance of variables, not that of formulae. Calculation of the importance measures is demonstrated by examples.

Let us first consider how to compute the Fussell-Vesely importance measure for variable *TS*. We need to know the probability that any prime implicant containing *TS* occurs. We note from table 3 that there are three prime implicants containing *TS*, namely prime implicants 3, 4, and 5. Further, we note that none of these prime implicants can occur at the same time, because either *TS* or *T* would yield two different values at the same time instance. Thus, the sought probability is given by

$$P\left(PI_3 \vee PI_4 \vee PI_5\right) = p_2 p_4 + p_2 p_5 + p_3 p_5 \tag{35}$$

and the Fussell-Vesely importance measure for TS is given by

$$DFMFV_{TS} = \frac{P\left(PI_3 \vee PI_4 \vee PI_5\right)}{R} \tag{36}$$

The Fussell-Vesely importance measure can be computed similarly for the other variables. The results are shown in Table 7.

For the Birnbaum importance measure, let us again consider the variable *TS*. For our purposes, it has three possible values (normal, low, null). Since there is only one time instance considered, we will find out which of these values produces the maximal risk and the minimal risk. For the maximization task, this happens by letting the probabilities $P(\varphi_4)$ and $P(\varphi_5)$ alternatively take the value 1 or their nominal value. We get the following possibilities:

TS	$P(\varphi_4)$	$P(\varphi_5)$	*R*(*TS*)
low	1	0	7,00E-03
null	0	1	7.13E-03
normal	0	0	2E-06

Table 6. The total risk when TS yields the values low, null, and normal.

It is easy to see that the maximal risk is obtained when *TS*=null surely (i.e. with probability 1), and the minimal risk when *TS*=normal. The Birnbaum importance of TS is then calculated by the subtraction 0,007131-2E-06=0,007129216.

Risk reduction worth and risk achievement worth are computed similarly.

Variable	F-V	BB	RRW	RAW
RP	3,36E-02	0,00711	7,78E-07	7,11E-03
T	1	0,00333	2,33E-05	3,31E-03
TS	9,15E-01	0,00713	2,14E-05	7,11E-03
MS	5,19E-02	0,00711	1,21E-06	7,11E-03

Table 7. Some probabilistic importance measures for the reactor control system model and top event (30)

Note that although the Birnbaum and RAW importances of *RP* equal that of *MS* at the precision used, in the results Birnbaum is slightly smaller for *MS* than *RP*, whereas RAW is slightly larger for *MS* than *RP*. The importances of *MS* and *RP* should be close to each other, since the two variables are in the same position structurally (see the prime implicants in Table 3); however, some variation between them should be expected since the probabilities of the respective formulae are different.

T is the most important variable by Fussell-Vesely and risk reduction worth, whereas TS is the most important variable by Birnbaum and risk achievement worth. *R* is less important than *MS* by all four probabilistic importance measures.

5 Independence of importance measures from the top event

As we have seen, the most useful importance measures depend on the top event considered. One of the primary advantages of DFM models, however, is the separation of the system model from the top event. Therefore it is natural to ponder how to formulate importance measures that are independent of the top event.

One way would be to define a new top event that would be a disjunction of a set of representative top events, and then proceed as normally with the calculation of the importance measures. This, however, might be computationally tedious due to the large number of prime implicants corresponding to the original top events.

Another way to accomplish a (near) independence of the importance measure from a top event is to combine the IMs for several top events. This combined IM would then represent the importance of a component in these top events. If the top events present, e.g., all the ways that a system can fail, the corresponding composite importance measure would describe the importance of a component to the system's failure in general. This would be advantageous in situations where several top events have already been analyzed and components importance measures been determined for some components of interest. One way to define such a composite importance measure is

$$\mathrm{IM} = \sum_{i=1}^{N} p_i^{top} IM_i \qquad (37)$$

Where IM is the composite importance measure over the set of top events, p_i^{top} is the probability of the top event i, N is the number of top events considered, and IM_i is the importance measure of the system with regard to the top event i. Thus each top event-dependent importance measure is weighted by the probability of its top event. If the importances of the top events in terms of their consequences differ, corresponding weights can be added in equation (37).

Other approaches are of course possible. For example, the Fussell-Vesely importance measure (section 3.2.1) could be extended so that in the nominator would be the number of all distinct prime implicants for any top event considered that contained the variable, and in the denominator the number of all distinct prime implicants for the top events considered.

6 Conclusions

We have seen that importance measures equivalent or analogical to Fussell-Vesely, Birnbaum, risk achievement worth and risk reduction worth can be formulated for DFM models. These analogues have the advantage that they can be used for arbitrary combinations of variables (e.g. groups). Model sensitivity can be applied to DFM only in a limited sense to the importance of states of variables. Application of importance measures developed for Markov models to DFM models is an open research question.

It would be advantageous to be able to evaluate the importance of components in a DFM model for the system's mission time. Therefore conceptual and mathematical work is needed in time-independent importance measures for DFM. Model sensitivity in the context of DFM models is clearly a research area where more work needs to be done, both conceptual and mathematical.

More research needs to be done also in the application of ideas from multistate system importance measures to DFM.

Explicit considerations of group importance measures [18] were left out of this report. However, they are as important in DFM as they are in classical reliability theory, and even more so: in a DFM model, several variables might correspond to a single component, for example the pressure and flow in a pipe. It is therefore of great practical importance that the importance measures defined in this report can directly be applied to groups of components.

Importance measures for multi-phase missions [30] in the context of DFM are also a promising research direction. However, research in modelling multi-phase missions with DFM is needed in general, and is a prerequisite to defining such importance measures.

There is a need for a practical study of importance measures in the DFM context. This could consist of applying the IMs developed in this paper to various DFM models and top events, and interpreting the results from the model's point of view.

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