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Constitutive modelling of fibre-reinforced brittle materials

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Abstract

The tensile behaviour of brittle materials can be enhanced by fibres. The cracking stress may increase, but the main influence is higher ductility after cracking. In some cases the material may also exhibit multiple cracking and tensile strain hardening behaviour thereafter.

A computational tool for analysing structures made of fibre-reinforced brittle materials was developed, starting from the micromechanical properties of fibres, matrix, and interface. The constitutive model is fairly general, including both single and multiple fracture, the fracture mode being either fibre rupture or pullout.

The study is divided into three parts: First, the mechanics of a single fibre is analysed. Second, the statistical tensile behaviour is evaluated by taking all fibre locations and orientations into account. Third, the macromechanical constitutive relation is incorporated into a finite element program for structural analysis.

The fundamental assumption in the single fibre analysis is the existence of a symmetry fibre within the matrix segment between cracks. This assumption enables the pullout analysis of a short fibre bridging several cracks. The strain energy of fibre and matrix, matrix fracture energy, fibre debonding energy, and frictional pullout energy are all included in the model. The theoretical value of the bond modulus was studied by means of the finite element method and dimensional analysis. Its value was found to be several orders of magnitude higher than the experimentally measured values reported in the literature.

The macromechanical behaviour is derived from several single fibre analyses by integrating over all possible fibre locations and orientations. The fibre orientation introduces additional phenomena: number of fibres bridging the crack, snubbing effect at the fibre bending point, and matrix spalling. The

snubbing effect increases the stress in the composite, while the other two have a decreasing effect. Matrix spalling considerably increases the crack width. The fundamental assumption is that the fibres carry axial stresses only. The result is a complete constitutive tensile relation of composite: stress–strain or stress–crack width curve, as well as a prediction of crack spacing. For the effects due to fibre orientation, a systematic experimental procedure should be developed.

The tensile model was extended into two and three dimensions by using the finite element method with smeared and discrete crack concepts and a multi-surface plasticity theory. The statistical tensile relation represents the behaviour normal to the crack. Automatic generation of interface elements on an existing geometry was developed to facilitate the modelling of discrete cracks after smeared crack analysis. The increase in the peak load of flexural members due to fibres, as measured in experimental tests, could be reproduced, which supports the validity of the approach proposed in this study.

Preface

The research reported in this thesis was carried out in two 12 month projects, the first of which took place at VTT Building Technology during 1993–1995 under the title "Constitutive modelling of fibre-reinforced hydraulic or ceramic materials" and was financed by the Finnish Academy and VTT Building Technology. The second project was performed in the Netherlands, at Delft University of Technology, Faculty of Civil Engineering and at TNO Building and Construction Research between February 1, 1996 and January 31, 1997, while the writer was on leave from VTT Building Technology. This European Community research training project, entitled "Constitutive modelling of fibre-reinforced brittle materials for numerical analysis" under the programme "Training and Mobility of Researchers", was financed by the European Commission. Additional financial support was received from the Herman and Saima Koivisto Memorial Fund, the Jenny and Antti Wihuri Foundation, the Foundation for Financial Aid at the Helsinki University of Technology, and CIMO. The support which made this research possible is deeply appreciated.

I am very grateful for the support and encouragement shown by my supervisor, Professor Mauri Määttänen, during the research. I would like to express my sincere gratitude to my hosts in Delft, Professor Johan Blaauwendraad at the Delft University of Technology, and Dr. Jan Rots at TNO Building and Construction Research.

I would like to thank all my colleagues at VTT Building Technology for their support and care. I am grateful to Professor Asko Sarja, manager of the Finnish Academy project, for his interest and belief in the research. It is my pleasure to thank Dr. Matti Pajari for his helpful comments. I would also like to thank VTT Building Technology for making my stay in Delft possible.

I am also grateful to all my colleagues at TNO Building and Construction Research, Numerical Mechanics Section, for providing good working conditions that enabled me to concentrate on the present investigation. I am especially indebted to Dr. Kalle Høiseth for his invaluable help as my contact person during my move to the Netherlands, and for his continued help since he returned to Norway. I also wish to thank Dr. Peter Feenstra and Mr. Rob ter Steeg for their invaluable discussions and help with the DIANA software. I am grateful to

Mr. Gerlach van Beinum and Mr. Jos Jansen for keeping the workstations and programs running smoothly. Thanks are also due to TNO Building and Construction Research for organising recreation for the family, including the Sinterklaas party and the Christmas presents.

Sincere thanks go to B.Sc. Adelaide Lönnberg for her proficient help with the English text of the present summary and the five papers.

Finally, I would like to express my deep gratitude to my wife Minna for her unending encouragement and patience during this work, and for her willingness to spend a year in the Netherlands with our small daughters Hilla and Varpu.

Espoo, July 1998

Jyrki Kullaa

List of publications

This dissertation consists of an introductory report and the following five papers:

I Kullaa, J. 1996. Dimensional analysis of bond modulus in fiber pullout. Journal of Structural Engineering, Vol. 122, No. 7, 783–787.

II Kullaa, J. 1996. Analysis of elastic fibre bridging in the multiple cracked composite. Journal of Materials Science, Vol. 31, 61–70.

III Kullaa, J. Micromechanics of multiple cracking. Part I: Fibre analysis. Accepted for publication in the Journal of Materials Science.

IV Kullaa, J. Micromechanics of multiple cracking. Part II: Statistical tensile behaviour. Accepted for publication in the Journal of Materials Science.

V Kullaa, J. 1997. Finite element modelling of fibre-reinforced brittle materials. Heron, Vol. 42, No. 2, 75–95.

Original features

This following features of this thesis are believed to be original:

1. A dimensional study to derive the theoretical value of the bond modulus, depending on the fibre diameter, fibre volume fraction, and moduli of elasticity of the fibre and the matrix.

2. Introduction of the fictitious two-fibre model resulting in a unified analysis of short fibre composites exhibiting single or multiple fracture.

3. The contribution of broken fibres in the multiply-cracked composite is taken into account.

4. Fibre distributions in 2- and 3-dimensional subspaces can be modelled.

5. Model of the effects of matrix spalling.

6. Model of hybrid composites based on geometric properties.

7. Automatic generation of discrete crack elements on an existing finite element model. The generation algorithm can also be used to model the ordinary reinforcement with interface elements representing the bond–slip relation.

Contents

Application of this publication are not included in the PDF-version.
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List of symbols

A	surface of fibre end distribution
A_f	area of fibre
A_f^i	area of fibre type *i*
A_m	area of matrix
A_m^i	area of matrix in pullout analysis
$\hat{A}_m^i$	area of matrix in calculation of composite stress
d	diameter of fibre
E_f	modulus of elasticity of fibre
E_m	modulus of elasticity of matrix
F_1, F_2	tensile force distributions in fibres
G_{db}	debonding energy of interface between fibre and matrix
G_{db}^i	debonding energy of interface between fibre and matrix before dynamic slip
h	characteristic length
$\hat{K}$	$\dfrac{\psi \kappa}{2 A_m E_m}$
l	segment length or fibre embedded length
l_f	fibre length
l_1, l_2	lengths of debonded zones
l_i^*	length of fibre end segment overlapping or not overlapping its symmetry fibre
P	$P_1 + P_2$, total load in the two-fibre model
P_1, P_2	forces in the fibres at the crack
$\hat{Q}$	$1+\dfrac{2 A_m E_m}{A_f E_f}$
S	slip between fibre and matrix
s	half of crack spacing
T	tensile force in matrix
t_f	frictional shear flow
u_i	dynamic slip of fibre within matrix segment
V_f	fibre volume fraction

V_f^i	fibre volume fraction of fibre type *i*
V_m	matrix volume fraction
w	crack width
w_{loc}	localised crack width
w_{ul}	closing crack width
x	co-ordinate along segment
Δ_i^l, Δ_i^r	relative displacement of fibre from left and right side of the crack *i*
ε_c	composite strain
ε_f	strain in fibre
ε_m	strain in matrix
$\overline{\varepsilon_m}$	average strain in matrix
κ	bond modulus
$\hat{\lambda}$	$\sqrt{\hat{K}\hat{Q}}$
λ_1	$\sqrt{\hat{K}(\hat{Q}-1)}$
λ_2	$\sqrt{\hat{K}(\hat{Q}+1)}$
μ	snubbing friction coefficient
ψ	perimeter of fibre
φ	fibre angle from perpendicular to crack due to matrix spalling
σ_c	composite stress
σ_f	stress in fibre at crack
$\overline{\sigma_f}$	average stress in fibre at crack
σ_f'	stress in fibre behind bending point
σ_m^a	aggregate bridging stress
θ	fibre orientation angle from perpendicular to crack
τ	shear stress between fibre and matrix
τ_f	frictional shear stress between fibre and matrix
τ_f^i	initial frictional shear stress between fibre and matrix
τ_u	bond strength of interface between fibre and matrix

1. Introduction

The tensile behaviour of brittle materials can be enhanced by fibres. Fibres can be added, for example, into concrete and other cementitious materials, ceramics, and brittle plastics. In this study only cementitious matrices are considered, but the results should also be valid for other brittle matrix composites.

There is an active search for new applications of fibre-reinforced cementitious composites. At present they are used e.g. in tunnel linings, pavements, and industrial floors because of better crack control. In addition, pipes, claddings, and thin roofing tiles have been reinforced with fibres. One application which has been widely studied is a structural beam with fibres as shear reinforcement instead of stirrups. Parallel to the search for new applications attempts have been made to enhance the properties of the composites. The enhanced properties usually include increased ductility and increased cracking stress. Recently interest has developed in tailoring strain-hardening composites (Li and Leung 1992, Maalej and Li 1995).

Developing new applications for fibre-reinforced brittle matrix composites often involves structural analysis. The structural engineer needs to know the material constitutive relation. Fibre-reinforced brittle materials exhibit highly non-linear behaviour due to cracking. Therefore, linear elastic analysis can only approximate the true behaviour. On the other hand, if the stress–crack opening relation is available the efficiency of the material can be better exploited. In material design it is most important to know what happens in the material and which parameters affect the constitutive behaviour.

Fibres used in brittle materials may be short or continuous, depending on the manufacturing process. Short fibres are gaining more interest because they can be added to a fresh mix, although the efficiency of continuous fibres is better. The advantage of short fibres is that they pull out from the matrix, leading to high ductility, contrary to continuous fibre composites which fail by fibre rupture. Fibre materials may include steel, glass, carbon, synthetic materials such as acrylic, aramid, polyester, polyethylene, polypropylene and nylon, and natural fibres such as cellulose, coconut, bagasse and sisal. In addition to fibre material and length, the fibre diameter and shape influence the composite behaviour by affecting the surface area and the bond respectively. Moreover, other fibre parameters affecting the composite action are the fibre volume fraction and fibre distribution. Different fibres also have different interaction with the matrix material.

Fibres in the mix distribute more or less randomly within the composite. Therefore the material can be considered homogeneous. The amount of fibres that can be added to the mix is quite low, the volume fraction typically being less than 3%. High fibre contents result in mixing and compaction problems and require special mixing or placing techniques. With low fibre contents the stiffening effect of the fibres before cracking is usually negligible. The fibres are effective mainly after cracking by bridging the cracks. They may also increase the cracking stress. In compression the effects of the fibres are usually negligible or the compressive strength may even decrease slightly.

When designing new composites the effects of different parameters should be known. The number of parameters can, however, be rather high as described above, requiring an expert to take them all into account. It is also impracticable to conduct experiments on all possible combinations of parameters. Therefore a computer program estimating the effects of different micromechanical parameters on the composite behaviour would be most useful in tailoring new materials.

The objective of this study is to develop a fairly general computational model to analyse fibre-reinforced brittle materials. The model should include both single and multiple fracture, the failure mode being fibre rupture or pullout. Moreover, the theory should be valid for low and high fibre contents, short or continuous fibres, different fibre distributions, and various fibre and matrix materials. Different energy terms should also be included. Energy is dissipated in matrix fracture and in fibre debonding and pullout. Also, the strain energy of the fibres and the matrix affects the behaviour of the composite.

This research is restricted to computing the constitutive behaviour under tensile loading, which represents the relation in the direction of the first principal stress. The compressive or shear behaviour is beyond the scope of this study. Matrix cracks are assumed to form perpendicular to the tensile load and extend over the cross-section. The cracking stress is assumed to be known. No fibre rupture or debonding is allowed before the first matrix crack. The fibres considered are discontinuous, straight and smooth with a negligible flexural and shear stiffness. The effects of residual stresses, radial compressive load, Poisson contraction, or temperature on the interaction between fibre and matrix are not considered in this study. In finite element analyses, only static monotonous loading is considered. The compressive stiffness is estimated using the rule of mixtures, and a small constant value for the shear stiffness is chosen. Fibres at the crack are assumed to be perpendicular to the crack in the finite element model, so that their pullout behaviour does not affect the shear stiffness of the crack.

Quite a number of analytical models exist for the fibre pullout problem (Budiansky et al. 1986, Lim et al. 1987, Wang et al. 1989, Hsueh 1990, Stang et al. 1990, Leung and Li 1991, Naaman et al. 1991, Leung 1992). These models are one-dimensional, with many common features. They include a shear-lag model in the full bond case, gradual debonding, and frictional sliding. In their study, Wang et al. (1989) considered only frictional bond. Lim et al. (1987) and Naaman et al. (1991) deduced the force distributions in the fibre and matrix, but derived an erroneous relative displacement at the fibre loaded end. Leung and Li (1991) and Leung (1992) developed a two-way debonding theory, in which the debonding can progress from both fibre ends. The pullout theories can be used to model the stress–crack width relation of composites failing by single fracture (Lim et al. 1987, Wang et al. 1989, Li et al. 1991, Li, V.C. et al. 1993, Kullaa 1994). In these models, fibres are analysed independently of surrounding fibres.

In the following, computational models for multiple cracking are briefly reviewed. The classical ACK theory by Aveston, Cooper and Kelly (1971) was first developed for composites with continuous aligned fibres, assuming a frictional bond. With the model, a tri-linear stress–strain curve can be produced, and the minimum crack spacing computed. The ACK theory was later extended by Aveston and Kelly (1973) taking into account elastic bond and partial debonding. Moreover, the crack spacing was derived for composites with non-parallel, continuous fibres assuming frictional bond only. Aveston et al. (1974) further extended the ACK theory by deriving the strength and crack spacing of composites with either continuous or short fibres in a random planar array assuming a frictional bond. Also in the theory by Kullaa (1994) a frictional bond was assumed for composites with short random fibres. With the theory, the non-linear stress–strain curve can be computed. A drawback of the model is that the force in the fibre can be totally transferred to the matrix. Moreover, crack localisation was not considered, nor the contribution of the broken fibres taken into account. Tjiptobroto and Hansen (1993a, 1993b) investigated the mechanism of strain hardening of composites with random discontinuous fibres and derived the critical fibre volume fraction and elastic strain limit using the energy changes associated with cracking. Li and Leung (1992) stated the conditions for steady-state and multiple cracking based on the fracture mechanics approach. Li, S.-H. et al. (1993) utilised the energy approach of fracture mechanics to explain the increased matrix cracking stress during multiple cracking of composites with continuous aligned fibres. Most of the existing theories cannot produce the non-linear stress–strain relation due to fibre pullout. Moreover, the contribution of broken fibres has not been taken into account. In addition, analysis of hybrid composites has not been reported.

In structural analysis it is difficult to model all fibres in the composite, although an attempt was recently made by Bolander Jr. and Saito (1997) who used a random geometry lattice model onto which each fibre was projected. Therefore the statistical tensile macromechanical relation is used to describe the material behaviour normal to the crack. If the relation is known, the structural behaviour can be estimated using the finite element method. Due to cracking and the non-linear behaviour thereafter, special algorithms are required. Theories of cracking have been developed for finite element analysis, including the smeared and discrete crack models (see e.g. Willam et al. 1987 and Rots 1988). Recently a multi-surface plasticity model was proposed by Feenstra (1993).

The study is divided into three successive parts: First, the crack bridging analysis of a single fibre is developed using the micromechanical material properties of fibres, matrix, and interface (App. I, II, III). Second, a statistical analysis is performed including all fibres with different locations and orientations or different fibre types (hybrid composite) to obtain the macromechanical tensile behaviour (App. IV). Third, the computed non-linear constitutive relation is incorporated into finite element analysis, representing the stress–strain relation normal to the crack (App. V).

2. Single fibre analysis

Composites failing in single fracture are typically modelled using the fibre pullout model (Figure 1a). In the pullout model, fibres are considered to be independent of the surrounding fibres. To the author's knowledge, a closed-form elastic solution of the pullout problem is not yet available. Therefore the problem is often approached with a one-dimensional model, in which the fibres and the matrix are assumed to carry tensile stress only whereas the shear stress is transferred by the interface between fibre and matrix.

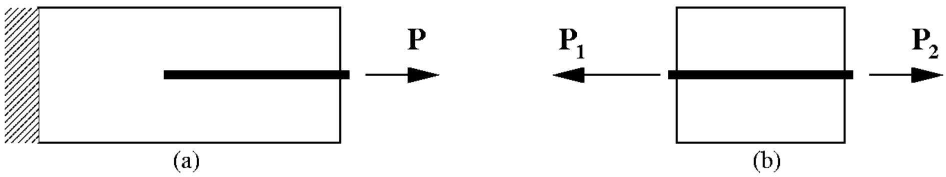

Figure 1. Fibre models in a) pullout, or single fracture, and b) multiple fracture.

The relative simplicity of the one-dimensional model with fibres being considered independent makes the approach most inviting. The assumption of independent fibres is justified if the stress gradient between adjacent fibres in a composite is small (Leung 1992). This approach enables analytical derivation of the average stress in the composite as a function of the crack width, if frictional shear stress is assumed (Li, V.C. et al. 1993). If full bond and gradual debonding are considered, the average stress can be numerically computed by analysing different fibre embedded lengths separately. Moreover, the effect of fibre distribution on the composite stress can be presented with a fibre orientation factor (e.g. Wang et al. 1989, Jain and Wetherhold 1992, Li, V.C. et al. 1993).

2.1 Two-fibre system

In multiply-cracked discontinuous fibre composites with a fibre bridging several cracks, the fibre forces at the adjacent cracks may differ and thus be out of balance (Figure 1b). It follows that the shear stress in the matrix halfway between fibres must be different from zero because of stress transfer from the fibre to surrounding fibres. This means that the fibre cannot be analysed independently, but the interaction must be taken into account. Therefore a fictitious two-fibre system is introduced, with two fibres being analysed simultaneously. Fibre 1 is the primary fibre, while fibre 2 represents surrounding fibres (Figure 2a). Therefore an average fibre force distribution for fibre 2 should be chosen. However, the average force distribution in the fibre is not known à priori. For the equilibrium condition, the choice of the force in fibre 2 at either crack is arbitrary. For simplicity, therefore, fibre 2 is idealised as having a force distribution which is a mirror image of that in fibre 1 (Figure 2b). The problem with two symmetrical fibres is relatively simple to analyse and satisfies the force equilibrium. However, the assumption of a symmetry fibre may lead to inaccuracy in relative fibre displacements if the force in fibre 2 is much different from the average. In the average sense the error presumably decreases when deriving the statistical macromechanical behaviour considering all fibre embedded lengths.

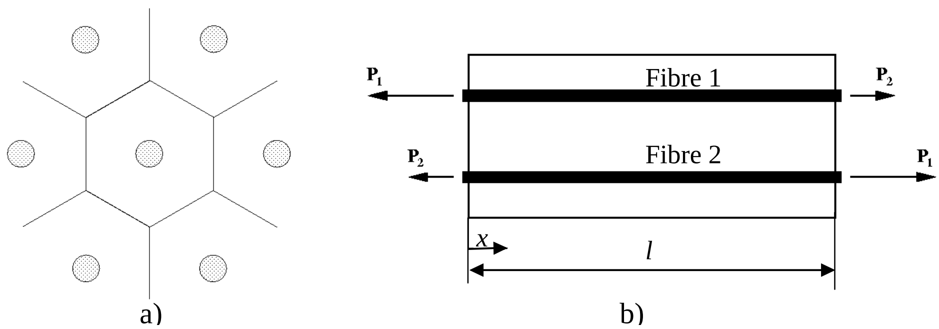

Figure 2. a) Hexagonal fibre arrangement and b) fictitious two-fibre system (App. III).

Let us study two adjacent fibres of the same type but mirror images of one another within a segment of length *l* so that the total force balance is satisfied at the crack. Using the free body diagram shown in Figure 2b, the equilibrium condition of the composite cross-section can be written:

$$F_1(x) + F_2(x) + 2T(x) = P \tag{1}$$

where $P=P_1+P_2$, the sum of the fibre forces at the crack, or the total load, $F_1(x)$ and $F_2(x)$ are the tensile forces in fibres 1 and 2, respectively, and $T(x)$ is an average tensile force in the matrix around one fibre. It should be noted that

$F_1(0)=F_2(l)=P_1$ and $F_1(l)=F_2(0)=P_2$. If the fibres and the matrix are linear elastic, obeying Hooke's law, the local strain difference between fibre 1 and the matrix can be derived using Equation 1 (App. II):

$$\varepsilon_{f1} - \varepsilon_m = \frac{1}{2A_m E_m}\left(\hat{Q}F_1 - P + F_2\right) \tag{2}$$

where

$$\hat{Q} = 1 + \frac{2A_m E_m}{A_f E_f}$$

A corresponding equation can be derived for ε_{f2}–ε_m. E_f and E_m are the moduli of elasticity of the fibre and the matrix, respectively, A_f and A_m are the cross-sectional areas of the fibre and the matrix, respectively, and ε_{f1}, ε_{f2}, and ε_m are the tensile strains in fibre 1, fibre 2 and the matrix, respectively. The constitutive equation of the fibre/matrix interface is still needed to obtain the relative displacements between the fibres and the matrix. Initially, in the full bond case, the bond–slip law is linear with a bond modulus κ (Aveston and Kelly 1973, Budiansky et al. 1986, Lim et al. 1987, Nammur and Naaman 1989, Leung and Li 1990, Stang et al. 1990, Naaman et al. 1991, Li and Chan 1994):

$$\tau = \kappa S \tag{3}$$

where τ is the interfacial shear stress and S is the slip, defined as the displacement of the fibre relative to the matrix halfway between two fibres, and can be derived from:

$$S = \frac{\tau}{\kappa} = \frac{1}{\kappa\psi}\frac{dF}{dx} \tag{4}$$

where ψ is the fibre perimeter. The force–displacement relation of the fibre at the crack can alternatively be derived using the finite element method. Combining the pullout model with the results from the finite element analysis, the bond modulus κ can be solved. Its value depends on the fibre diameter, fibre volume fraction, and moduli of elasticity of the fibre and the matrix. Therefore a

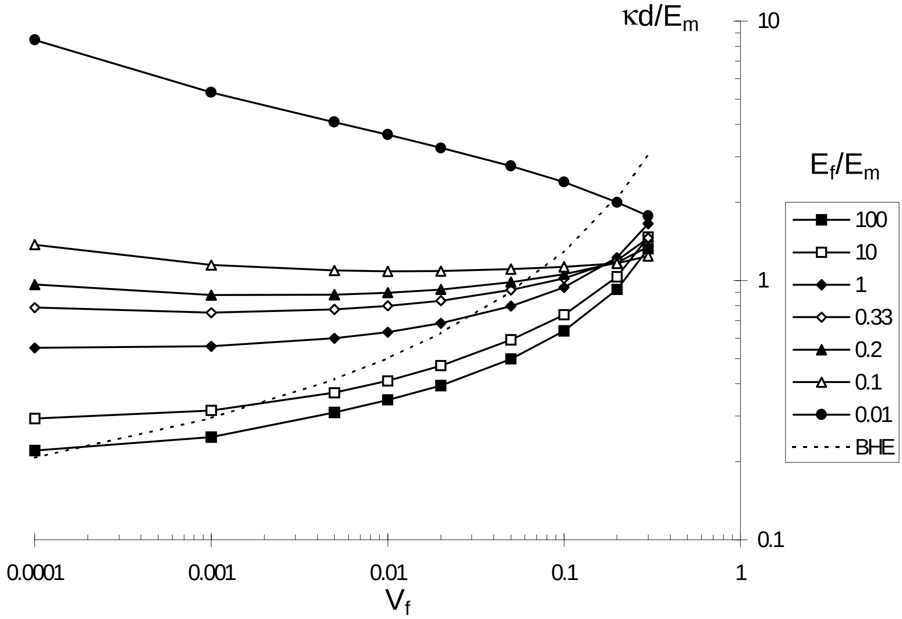

Figure 3. Normalised bond modulus versus fibre volume fraction with different fibre and matrix moduli ratios. The dashed line is from Budiansky et al. (1986) with d=1 mm, E_m=21 GPa, and l=30 mm (App. I).

parametric study was performed using the dimensional analysis (App. I). The result was a parametric plot, shown in Figure 3, from which the value of κ can be read.

Debonding is assumed to initialise if the interface shear stress exceeds the bond strength as proposed by Naaman et al. (1991), Lim et al. (1987), and Leung and Li (1991). Stang et al. (1990) and Leung (1992) also proposed a fracture-based debonding theory, and compared the two theories. They found that the analyses were identical provided the effective shear strength was defined separately for both theories. From the shear stress distribution in fibre pullout, it was shown that debonding can proceed both from the loaded and from the embedded end (Leung and Li 1991). This two-way debonding is also included in this study. After debonding the interfacial shear stress is assumed to be a constant frictional shear stress τ_f. Only after the fibre is fully debonded within the segment is the debonding energy or the slip-dependent friction taken into account with a decaying frictional shear stress. This approach is discussed in Section 2.4.

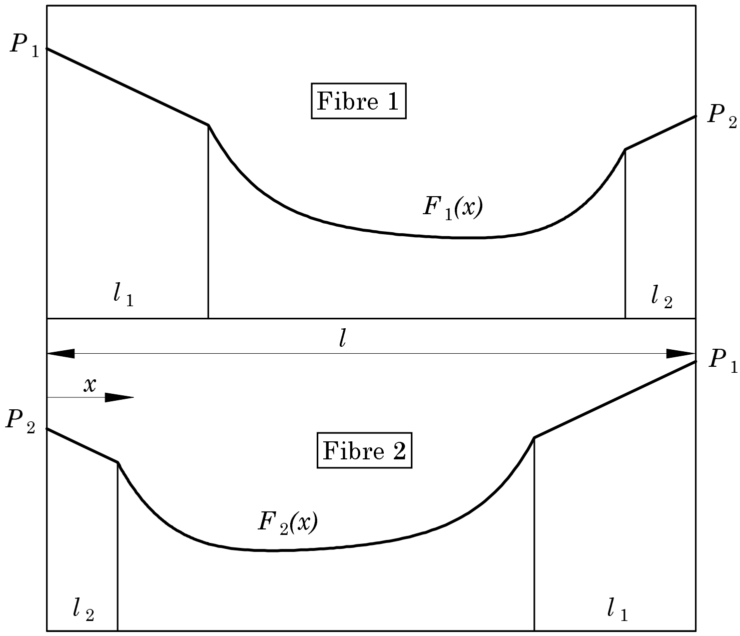

Figure 4. Distributions of fibre forces where both ends are debonded less than in mid-segment (App. III).

In the two-fibre system with a symmetry fibre there are three possible cases for the fibre force distribution in every cross-section, depending on the bond (Figure 4 is used as an example):

- Full bond in both fibres ($l_1 < x < l - l_1$):

$$F_1(x) = Ae^{-\lambda_1 x} + Be^{\lambda_1 x} + Ce^{-\lambda_2 x} + De^{\lambda_2 x} + \frac{P}{\hat{Q}+1} \tag{5}$$

- Full bond in fibre 1, fibre 2 debonded ($l - l_1 < x < l - l_2$):

$$F_1(x) = Ge^{\hat{\lambda} x} + He^{-\hat{\lambda} x} + \frac{1}{\hat{Q}}\left[P - F_2(x)\right] \tag{6}$$

- Frictional bond in fibre 1:

$$F_1(x) = \begin{cases} P_1 + t_f\, x, & 0 < x < l_1 \\ P_2 + t_f\,(x - l), & l - l_2 < x < l \end{cases} \tag{7}$$

where

$$\lambda_1 = \sqrt{\hat{K}(\hat{Q}-1)}$$
$$\lambda_2 = \sqrt{\hat{K}(\hat{Q}+1)}$$
$$\hat{\lambda} = \sqrt{\hat{K}\hat{Q}}$$
$$\hat{K} = \frac{\psi\,\kappa}{2A_m E_m}$$

and t_f is the frictional shear flow ($t_f = \tau_f \psi$), the sign of which depends on the direction of friction. Corresponding force distributions can be written for fibre 2. The coefficients *A*, *B*, *C*, *D*, *G*, and *H* are linear functions of P_1 and P_2 and can be determined from the symmetry and continuity conditions of the fibre force.

2.2 System equations

In order to apply the two-fibre theory to the analysis of composite materials, the forces P_i in the fibre at the cracks (Figure 5) should be derived from the crack widths. Therefore a compatibility condition is needed, which is stated as mutually equal crack openings in the strain hardening stage, whereas in the softening stage one crack opens while the others close:

$$\Delta_i^l - \Delta_i^r = w_i \tag{8}$$

where Δ_i^l and Δ_i^r are the relative displacements of the fibre on the left and right side of crack *i*, respectively, and w_i is the width of crack *i*. For example, Δ_i^l for fibre 1 in Figure 4 is (App. III):

$$\Delta_i^l = S\big|_{x=l-l_2} + \int_{l-l_2}^{l} (\varepsilon_{f1} - \varepsilon_m)\, dx \tag{9}$$

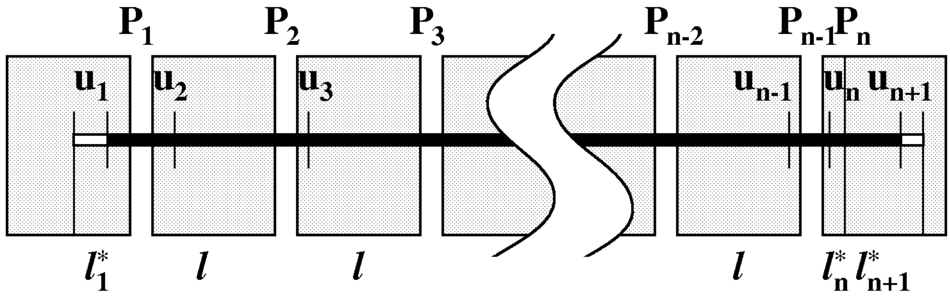

Figure 5. Model of crack bridging (App. III).

The first and second terms on the right hand side of Equation 9 can be obtained using Equations 4 and 2, respectively.

In case of a dynamic mechanism of fibre pullout, the rigid body displacements u_i within the matrix segments are also unknown (Figure 5). For example, if fibre 1 slips to the right in Figure 4, it follows that $l_2=l$. The first term $S|_{x=0}$ in Equation 9 is the unknown u_i for which Equation 4 cannot be used since the fibre is debonded. An additional equation is therefore needed, which is derived from the relation between fibre forces at adjacent cracks:

$$P_i = P_{i-1} + t_f\, l \qquad (10)$$

Equation 8 is a system of equations from which the unknown variables P_i and u_i can be solved. Thereafter the distributions of the stress in the fibre and the matrix and the shear stress of the fibre/matrix interface can be derived and the debonding criterion tested.

The analysis of fibre ends leads generally to an infinite number of equations, because when using symmetry fibre idealisation the fibre force at the crack next to the fibre end depends also on the subsequent fibre. To overcome this problem, a simplified approach has been developed in which the fibre segments not overlapping their symmetry fibre (l_1^* and l_n^* in Figure 5) are analysed with the single fibre theory. This approach is justified because the two adjacent fibre regions are pulled out in the same direction, which is the case also in the single fracture. It may, however, cause inaccuracy compared with the two-fibre system, the error increasing as the moduli ratio of the fibre and the matrix or the fibre volume fraction increases.

The proposed model enables both single and multiple fracture. In fact, also in the multiply-cracked case fibres with a high inclination angle bridge a single crack only. Analysis of these fibres is similar to the pullout theory in single fracture, if the fibres extend less than halfway between the cracks. In the ACK theory put forward by Aveston et al. (1971), only sudden failure due to fibre rupture can be analysed. The two-fibre system makes it possible to analyse both fibre rupture and pullout. After fibre rupture the contribution of broken fibres can also be taken into account. In addition, hybrid composites can be modelled.

2.3 Hybrid composite

If hybrid composites with two or more fibre types are analysed, two different definitions of the matrix cross-sectional area should be distinguished. In fibre analysis, the matrix area A^i_m occupied by a single fibre of type i is:

$$A_m^i = \frac{V_m}{V_f} A_f^i \tag{11}$$

where V_f and V_m are the volume fractions of all fibre types and the matrix, respectively, and A^i_f is the cross-sectional area of fibre type i. However, if the tensile stress in the matrix is computed, the force transferred from a single fibre of type i must be divided by a different matrix area:

$$\hat{A}_m^i = \frac{V_m}{V_f^i} A_f^i \tag{12}$$

where V_f^i is the volume fraction of fibre type i. The areas A_m^i and $\hat{A}_m^i$ in Equations 11 and 12 are equal if a single fibre type is analysed. It is also worth noting that the composite with a single fibre type can also be analysed as a hybrid composite by dividing the fibres into two or more groups.

2.4 Energy terms in fibre analysis

Tjiptobroto and Hansen (1993a) studied the strain-hardening mechanism in concrete reinforced with discontinuous fibres using an energy approach. They considered the following energy changes during cracking: matrix fracture energy, fibre debonding and frictional pullout energy, and the strain energy of the fibres and the matrix. In the following, different energy terms are discussed, including how they are taken into account in the present model.

The elastic strain energy of the fibre and the matrix is taken into account by Hooke's law in the pullout theory. The constant interface frictional energy during pullout is also considered. The debonding energy can be taken approximately into account with a decaying frictional stress. Ba•ant and Desmorat (1994) suggested a softening relation for debonding, similar to the tension softening for concrete. The debonding energy is defined as the area below the interfacial shear stress–relative slip curve. The bond–slip relation used in this study is very similar to that of Naaman et al. (1991), allowing an instantaneous drop of shear stress from bond strength τ_u to the initial frictional shear stress τ^i_f (Figure 6). The frictional shear stress is then held constant within the matrix segment during the progressive debonding process. Thereafter arbitrary dependence of the frictional shear stress on dynamic slip can be assigned. The decreasing friction due to mechanical abrasion can be modelled with the same method (Naaman et al. 1991). It should be noted that the basic physical concepts and magnitude of slip are very different in the two models. The breakdown mechanism is directly associated with the fracture energy of the interface with a high strength degradation rate in the breakdown zone. The rate of the slip-dependent friction is lower and related to the damage suffered by the fibre or the matrix during frictional sliding (Li and Stang 1997).

The limitations of this approximate modelling of the debonding energy must be studied. It is done in the following by investigating the debonding energy at the fibre loaded end during progressive debonding in fibre pullout. According to Wang et. al (1989) the displacement of the fibre loaded end at the onset of pullout is

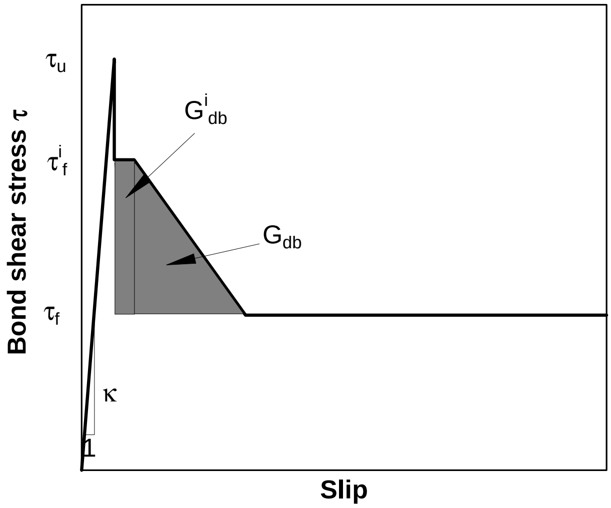

Figure 6. Assumed bond–slip relation between fibre and matrix. The shaded areas represent the total debonding energy and the debonding energy dissipated during the progressive debonding process.

$$S = \frac{2\tau_f^i l^2}{E_f d} \tag{13}$$

where d is the diameter of the fibre and l is the fibre embedded length. In Equation 13 the contribution of the matrix is neglected. If the initial shear stress τ_f^i is constant, the debonding energy dissipated at the fibre loaded end at the onset of dynamic slip is

$$G_{db}^i = (\tau_f^i - \tau_f)S \tag{14}$$

which is represented by the hatched area in Figure 6. The approximate slip-weakening concept is valid if the debonding energy in Equation 14 can be considered small compared to the total debonding energy (light shaded area in Figure 6). From Equations 13 and 14 it can be seen that this is the case for low values of fibre embedded length or crack spacing, and for low $\tau_f^i - \tau_f$ and high fibre modulus and diameter. Consider, for example, the FR-DSP material (App. V), with τ_f^i=6 MPa, τ_f=4 MPa, l=3 mm, E_f=210 GPa, d=0.15 mm. The debonding energy before dynamic slip derived from Equations 13 and 14 is

G^{i}_{db}=6.9 N/m, which is only 6% of the reported G_{db}=120 N/m. In this case there must be a large transition zone between the undebonded and debonded regions (Leung and Li 1991) and the model can be used without significant error.

The matrix fracture energy release is modelled using the fictitious crack model by Hillerborg et al. (1976). The energy is defined as the area below the tension-softening curve ($\sigma_m^a - w$ curve), which is usually a decaying curve. In this study the relation is generated using the theory of Tran Tu and Kasperkiewicz (1994). The additional stress $V_m \sigma_m^a$ and strain $V_m \sigma_m^a / E_m$ are then added to the pullout relation similarly to the aggregate bridging action in the study by Li, V.C. et al. (1993). Matrix fracture energy also affects the cracking stress of the composite (Aveston et al. 1971, Li and Leung 1992, Tjiptobroto and Hansen 1993b).

3. Macromechanical tensile behaviour

For structural analysis the behaviour of a single fibre is not very useful, whereas the macromechanical constitutive relation of the composite is of greater interest. The objective of this chapter is to derive the statistical tensile stress–strain relationship of brittle matrix composites for numerical analysis. The macromechanical behaviour can be obtained by considering a representative volume element (RVE) with a fibre volume fraction V_f in every cross-section. The cracks are assumed to extend over the whole cross-section, crossed by randomly distributed fibres.

The main assumptions and restrictions concerning the fibres are that they are separate, straight, and smooth. Moreover, in case of inclined fibres, they are completely flexible carrying axial stress only. If deformed or bundled fibres are analysed, the bond properties or fibre diameter can be adjusted respectively in order to approximate the pullout behaviour, but the model should be used with caution. If the bending or shear energy of the fibre is significant, the model may lead to erroneous results by neglecting the flexural and shear stiffness or by failing to predict the fibre rupture due to bending.

The tensile stress σ_c in the cracked composite can be evaluated from the average fibre tensile stress $\overline{\sigma_f}$ at the crack in the direction normal to the crack:

$$\sigma_c = V_f \, \overline{\sigma_f} \tag{15}$$

Consider a case where the embedded ends of fibres at the crack are uniformly distributed over surface A. The average fibre stress can be obtained by integration over all fibre locations l and orientations θ:

$$\overline{\sigma_f} = \frac{1}{A}\frac{2}{l_f}\int_0^{l_f/2}\int_A \sigma'_f\,(l,\theta)\,e^{\mu(\theta-\varphi)}\cos\varphi\cos\theta\,dA\,dl \tag{16}$$

where A depends on the fibre distribution. If the distance between cracks is $2s$, σ'_f is the stress in the fibre at the crack obtained from the single fibre analysis with crack spacing $2s/\cos\theta$. The crack width in the fibre analysis is the total crack width minus the additional crack opening due to matrix spalling.

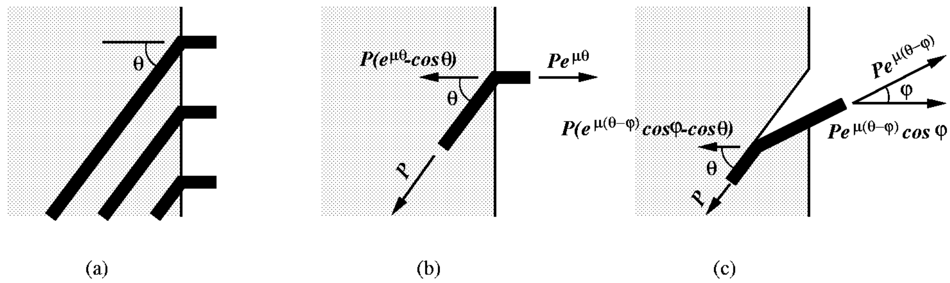

Figure 7. Effects of fibre orientation: a) number of fibres at crack, b) pulley effect, and c) matrix spalling.

In Equation 16 three effects due to fibre orientation are taken into account (Figure 7). First, Wang et al. (1989) showed that the number of fibres bridging the crack is proportional to cosθ (Figure 7a), which is always less than or equal to one, decreasing the average fibre stress and the stress in the composite.

Second, fibre snubbing is modelled by a frictional pulley at the fibre bending point, μ being the snubbing friction coefficient (Li, V.C. et al. 1993). The stress in the fibre at the crack, according to the classical belt friction theory, is then $\sigma_f' e^{\mu\theta}$, if it is σ_f' immediately behind the bending point (Figure 7b). Because the term $e^{\mu\theta}$ is always greater than or equal to one, the snubbing effect increases the fibre stress and the stress in the composite. The increased fibre stress may also cause fibre rupture, further cracking, or matrix spalling. It should be noted that the pulley effect is independent of the curvature radius, which makes the pulley model particularly attractive because the radius is generally not known.

Matrix spalling is caused by local loads at the fibre bending point. As a result, the matrix may split under the indentation load and form a wedge spall (Li 1994). Matrix spalling changes the geometry at the crack, allowing an inclined position of the fibre at the crack (Bartos and Duris 1994) with an inclination angle φ (Figure 7c). Because of the altered geometry the fibre has a shortcut, causing additional crack opening. Therefore the crack width and the composite strain increase. Because of a lower curvature angle, the additional stress due to the pulley effect decreases. Moreover, provided the fibre acts like a rope, the stress component normal to the crack is lower than the tensile stress in the fibre.

Due to these two effects the stress in the composite decreases, which may also allow survival of the fibre from rupture.

Independently of and since the present study, a fairly similar theory of fibre orientation effects has been presented by Leung and Ybanez (1997) in which the elongation of the free part of the fibre is also considered. However, its contribution to the crack width is usually insignificant with relatively low spalled lengths.

The results obtained with the proposed model agree well with the ACK theory. Using the present theory, the complete stress–strain curve can be computed regardless of the fibre length, orientation, failure mode, or number of fibre types. Therefore the proposed model can be considered an extension of the ACK model.

In the analysis of a strain-hardening engineered cementitious composite (ECC) with randomly oriented discontinuous polyethylene fibres, the snubbing friction coefficient and matrix spalled length were introduced, which are not included in the ACK theory. Reasonable agreement with the experimental results was obtained (App. IV). However, the analysis resulted in too low a crack spacing, which was found to be fairly sensitive to the estimated spalled length.

4. Finite element modelling

With finite element modelling the constitutive tensile behaviour of fibre-reinforced brittle materials can be extended into two or three dimensions to analyse complex structures under general loading. The idea of a fictitious crack model by Hillerborg et al. (1976) describing the crack propagation with a tensile softening relation makes it possible to analyse complex cracking structures with the finite element method. The relation can be used e.g. in smeared or discrete crack models. These models have been widely used for plain concrete with a decaying tensile softening relation. In addition, a combined plasticity model can be used, with separate yield conditions in tension and compression regimes. These approaches were used in this study for fibre-reinforced cementitious composites (FRC), with the sole exception that the stress–crack opening relation is different from that of plain concrete. The tensile behaviour of FRC is not invariably descending but may also have ascending regions. This is not, however, in contradiction to the methods; the approach is justified since cracking is the main phenomenon for both materials.

Initially, the positions of cracks in the structure are not known. Rots (1988) suggested that a predictor analysis be made using smeared crack analysis to obtain the locations and orientations of the main cracks (Figure 8a). Thereafter a corrector analysis is performed using the discrete crack approach with the cracks being modelled with interface elements. The geometry must therefore be modelled twice, which may delay the analysis process. In this study a procedure is developed to automatically generate the interface elements on the existing geometry. The geometry of the cracks can be created using line objects (Figure 8b). Although the plane elements adjacent to the interface elements may become distorted, the method removes the obstacle of re-modelling the structure (Figure 8c). The generation algorithm can also be used to model the ordinary reinforcement with interface elements representing the bond–slip relation. Moreover, holes, outer boundary, or other details of the structure can also be generated (Kullaa and Klinge 1995).

The definition of tensile strain for smeared crack analysis differs depending on the fracture mode. If the composite fails in single fracture, the tension softening curve can be used in the same manner as for plain concrete (Ba•ant and Oh 1983, Rots 1988). The strain consists of the elastic strain of the intact material a)

b)

c)

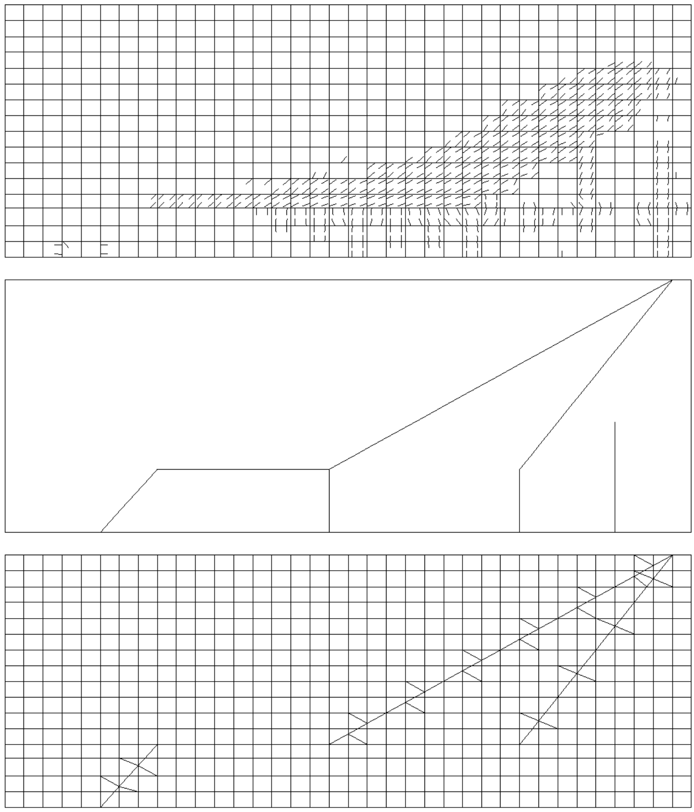

Figure 8. Automatic mesh refinement from smeared crack model to discrete crack model. a) Crack pattern in smeared crack analysis, b) position of discrete cracks, and c) discrete cracks modelled on the existing geometry.

and the crack width *w* divided by a characteristic length *h*: $\varepsilon_c = \varepsilon_m + w/h$. It should be noted that *h* is not a material property but is related to the element size.

In the case of a strain-hardening composite failing in multiple fracture, scaling according to element size should be executed only for those terms comprising the characteristic length *h*. At the strain hardening stage, the strain consists of the average matrix strain and the crack width: $\varepsilon_c = \varepsilon_m + w/2s$, both components of which can be interpreted as true strains in the smeared crack analysis. The strain no longer depends on the element size but on the crack spacing *2s*, being thus a material property.

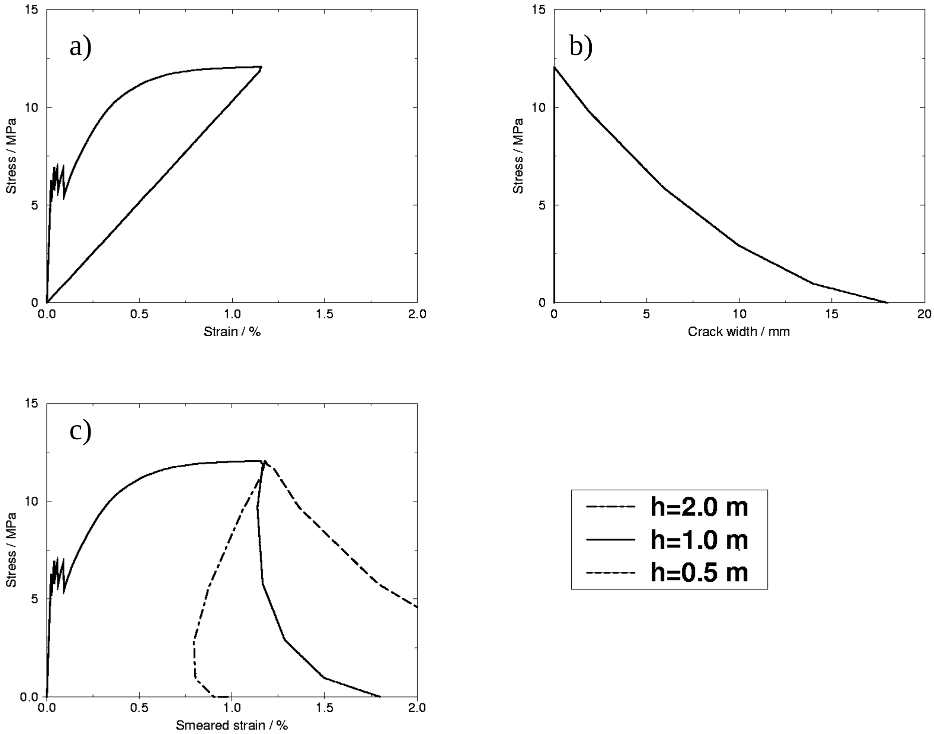

Figure 9. Strain components in a multiply-cracked composite with different values of characteristic length h. a) Stress–strain; b) stress–crack width; c) stress–smeared strain relations (App. V).

During crack localisation in the multiple cracking stage, the strain consists of three terms: the average matrix strain, the closing crack widths w_{ul} equal for all cracks, and the additional crack width of the opening crack w_{loc}: $\varepsilon_c = \varepsilon_m + w_{ul}/2s + w_{loc}/h$. When using the smeared crack concept, it must be noted that only part of the crack width depends on the characteristic length h, while the other part depends on the crack spacing. The components of the strain with different values of h are illustrated in Figure 9. At the strain hardening stage the strain is seen to be independent of the characteristic length, whereas during crack localisation it is strongly element-related. It should be noted that the unloading pattern of the closing cracks also has an effect on the tensile softening relation. In Figure 9 secant unloading is presumed.

Structures analysed in the present study included steel fibre-reinforced beams containing different fibre volume fractions and exhibiting multiple fracture; kevlar fibre beams of different sizes and fibre contents without ordinary reinforcement; a reinforced beam where kevlar fibres replaced the stirrups; and a concrete pipe under a diametral line load, either without fibres or reinforced with Wirex or Dramix fibres.

The analyses were performed using the cracking models in the DIANA finite element package. A detailed description of the analyses is given in App. V. A distinct increase in the peak load due to fibres could be produced. Moreover, the computations gave a similar size effect of beams to that observed in the experiments. The correct tensile relation proved essential for predicting correctly the flexural behaviour. In particular, the shape of the post-peak tensile relation has a significant influence on the load–deflection behaviour. The smeared and discrete cracking models were seen to work nicely with these applications. With more complicated structures, however, other problems may arise. The reinforced beams with fibres replacing the stirrups introduced difficulties e.g. in the convergence and modelling of the dowel action of the reinforcement. The increase of peak load due to fibres could nonetheless be produced. However, such analysis is still beyond the realm of engineering practice and cannot be used to verify the computed tensile relation.

5. Discussion

There is in retrospect much to be gleaned from the various parts of the research, just as there are subjects open to further study. These are discussed below.

5.1 Fibre analysis

The advantages of the two-fibre system include the ability to model discontinuous fibres and their pullout in the multiply-cracked stage. Fibre 2 is defined as a symmetry fibre, although in reality it should represent the average surrounding fibre. The choice of the symmetry fibre is justified by the average decrease in error when analysing all fibre embedded lengths. The symmetry fibre approach also makes it possible to extend the theory to hybrid composites. The error due to the symmetry fibre assumption may, however, be significant if the properties between each fibre type vary considerably.

The comparisons between the single- and two-fibre models (App. II) are somewhat questionable, because the single-fibre model does not satisfy the equilibrium conditions and cannot therefore be used for a multiply-cracked composite. In case of multiple cracking the two-fibre theory should be preferred even with a low accuracy requirement, since the relative error of the single fibre system was studied in the full bond stage only. Applying the single fibre theory may fail to predict the macromechanical behaviour and therefore cause larger errors than those reported.

One difficulty with short fibres is due to the fibre ends. The simplified approach used for fibre ends was shown to be liable to inaccuracy, restricting the validity of the model to low fibre contents or moduli ratios. This is not a problem with typical fibre reinforced cementitious composites but may cause inaccuracy with some ceramic composites with high fibre contents.

The value of the bond modulus was determined by the finite element method for fibre pullout. The analysed values were estimated to be pertinent also to the two-fibre theory. However, whether the bond modulus is of same magnitude also for a two-fibre system was not studied.

The system equations are based on the deformation theory, meaning that the unloading path is roughly the same as the loading path in fibre pullout. This may cause inaccuracy, which is probably negligible in a single fracture after the peak, because the crack opening due to pullout of the shorter embedded end of the fibre far exceeds the displacement of the longer embedded length. However, if the interface exhibits slip-hardening behaviour, the permanent crack opening may be significant at the onset of unloading, the significance decreasing with an increasing pullout length. In this study only interfaces with slip-softening behaviour were considered. In multiple fracture the error can be minimised by defining a permanent crack opening for the closing cracks. Incremental equations should nonetheless be developed if unloading is to be more thoroughly investigated.

The tensile softening relation due to the matrix fracture energy is added to the computed average stress in the composite. Strictly speaking, the stress in the matrix at the crack should already be taken into account in the boundary conditions of the two-fibre theory. However, contribution of the matrix is only significant at low crack widths, whereas fibres contribute up to a crack width of half the fibre length. It is also believed that the error in the stress–strain relation of the composite is not significant if the matrix contribution at the crack is only later considered.

The slip-weakening behaviour of the interface due to debonding is only taken into account after the fibre has fully debonded within the matrix segment. The frictional shear stress thereafter depends on the rigid body displacement, or dynamic slip of the fibre within the segment. A more correct way would be to take the varying shear stress already into account during the debonding process. The size effect in the fibre pullout could then be studied. Nevertheless the debonding energy was successfully considered with the simplified approach for the material called “fibre-reinforced densified small particles” (FR-DSP) (App. V).

In the analysis of hybrid composites the area of the matrix is computed using the geometric properties only. Presumably the stiffness ratio of the fibres also affects the matrix area occupied by a single fibre. Consider, for example, a hybrid composite with two types of fibre of similar geometries and volume fractions, but different moduli of elasticity. The area of the matrix occupied by the stiffer fibre is probably larger than that of the flexible fibre. This is not, however, taken into account in the proposed model.

5.2 Macromechanical model

In the statistical analysis some simplifications were adopted. First, once new cracks form, the computing is started from the beginning and the debonded lengths are zeroed. Second, the cracks are assumed to be of equal width and spacing. Third, cracking takes place at constant average stress in the matrix. The latter assumption may result in too low a crack spacing and multiple cracking process to occur at a constant, instead of at an ascending stress of the composite. These computational simplifications are not believed to cause major error in the subsequent structural analysis.

A systematic experimental procedure should be developed to measure and distinguish the effects of fibre orientation. In the most analyses in this study the pulley effect and matrix spalled length were estimated as they were not available. Moreover, the matrix spalling criterion should be included in the proposed model. Due to the estimated parameters, the validity of the proposed methods for modelling the two phenomena cannot be assessed.

The macromechanical parameters are computed from the average stresses and strains, which are approximated by numerical integration. In this study the rectangular rule of integration was used, which is very simple but the accuracy of which must be checked. The general accuracy is difficult to assess because it is affected by the load level, crack spacing, fibre distribution, pulley effect, matrix spalling, and possible rupture of fibres. The accuracy was estimated using examples in App. IV, section 3.1. First, in single fracture mode, the relative error of the orientation factor, for which the exact solution is available, was studied varying the number of inclination angles. Using 5 and 7 angles, the relative error was 1.6% and 0.8%, respectively. Equal accuracy was obtained using a snubbing friction coefficient μ=0 and μ=0.25. For two-dimensional fibre distribution the error was more than three times lower. The error in average stress in the fibre was estimated by varying the number of embedded lengths and comparing the result with that obtained with 100 embedded lengths. With 5 and 7 integration points the relative error was 0.5% and 0.12%, respectively.

Second, in a multiply-cracked composite, with 5 and 7 orientation angles the relative errors of the stress in the composite were 1.4% and 1.02%, respectively, compared to the stress obtained with 100 angles. Using 5 and 7 fibre embedded lengths led to 0.58% and 0.24% relative error, respectively, compared with using 100 embedded lengths. However, in case of fibre pull-out, with such a small number of integration points the error of the strain in the composite at the peak was much higher, exceeding 10%. The error in the peak stress was nevertheless negligible.

More sophisticated integration rules could also be used. Simpson's rule of integration or the Romberg algorithm could be more efficient than the rectangular rule (e.g. Cheney and Kincaid 1994). Gaussian integration formulae are also accurate and could be used for variable θ. However, for variable l, integrations between both 0 and $l_f/2$, and between s and $s+l_f/2$ must be performed (App. IV). With Gaussian integration formulae the values of the integrand should be evaluated separately at different points depending on the integration limits.

In practical computation the number of cracks per fibre is limited due to the increasing computational time. If the fibre bridges more than 32 cracks, the computation may take too long depending on the number of integration points. Normally the limit of 32 cracks per fibre is not exceeded except with continuous fibres, in which case another program can be used with the assumption of equal forces in the fibres at adjacent cracks. Then only one segment between two cracks need be considered and no integration in respect of fibre embedded length need be performed. This implies that anchorage at the fibre ends is provided and that the fracture mode is invariably fibre rupture.

5.3 Structural analysis

The finite element method was applied to flexural members under proportional loading. The experimental behaviour could be reproduced, which justifies utilisation of the used cracking models. For non-proportional loading the analysis would involve further aspects, such as unloading and the crack shear stiffness. Also, in shear-critical problems, e.g. when using fibres as shear reinforcement, the effect of the shear stiffness of the crack should be studied. In complex problems with several cracks and more than one equilibrium path, automatic incremental algorithms should be further developed. Moreover, the dowel effect of the main reinforcement should be investigated.

Finite element applications concerned problems in two dimensions only. The method is, however, also valid for three-dimensional problems. It should be noted that the constitutive model in most problems was still computed using a three-dimensional fibre distribution. This is a significant advantage as fibres are usually distributed in three dimensional array, but two-dimensional finite element analysis is sufficient for most problems. Finite element applications in three dimensions were left for future research.

6. Conclusions

The objective of this research was to develop a computational tool for analysing fibre-reinforced brittle matrix composite materials and structures. The main conclusions are listed below.

1. The micromechanical model is available for material design and analysis. The model can predict the effects of several micromechanical parameters on the tensile behaviour of the composite. The model can be used e.g. in tailoring strain-hardening materials. In addition, the macromechanical tensile constitutive behaviour can be used in structural analysis.

2. The two-fibre theory makes it possible to analyse the multiple fracture of brittle matrix composites reinforced with discontinuous fibres. With the model the fracture mode of fibre rupture or pullout can be analysed. To the author's knowledge this is the first micromechanical model to produce the non-linear stress–strain relation of multiply-cracked discontinuous fibre composites.

3. The analysis of hybrid composites with two or more fibre types is mathematically quite straightforward.

4. The contribution of broken fibres is considered in the model.

5. Fibre ends are analysed with a simplified model, which may cause inaccuracy in the results if the fibre volume fraction and the ratio of the elastic moduli of the fibre and the matrix are high (V_f >30%, E_f/E_m>100). In typical cementitious composites these limits are not exceeded. The approach of analysing fibres independently in the single fracture mode is also justified within these limits.

6. The theoretical bond modulus can be read from a curve if the fibre volume fraction, fibre diameter, and the elastic moduli of the fibre and the matrix are known.

7. The basis for computing the macromechanical tensile behaviour is two-fibre analysis. The effects of fibre orientation are taken into account with simple

models, which do not affect the fibre analysis. Therefore the theory is fairly easy to program. The number of fibres bridging the crack is smaller due to fibre orientation, thus decreasing the stress in the composite, whereas the frictional pulley effect does the opposite. Matrix spalling reduces the pulley effect and increases the crack opening. When analysing a strain-hardening engineered cementitious composite (ECC) in tension, it was noticed that matrix spalling alone could explain the extremely high strain capacity. The models of the fibre orientation effects are yet to be verified against measured properties, some of which were unavailable in this study.

8. Several energy terms are included in the constitutive model. In addition to the strain energy of the fibres and the matrix, the matrix fracture energy, fibre debonding energy, and frictional pullout energy are also considered. Including the fibre debonding energy produces a decaying post-peak relation observed in the experiments.

9. The present model agrees well with the ACK theory and its extensions. With the ACK model the complete stress–strain curve can only be computed for continuous, aligned fibres. The present model is therefore an extension to the ACK theory since it can produce the constitutive relation independent of fibre length, orientation, failure mode, or number of fibre types.

10. In two-dimensional finite element analysis the enhancing effect of the fibres could be reproduced by using the crack normal stress–crack opening relation, to which the computed macromechanical constitutive tensile behaviour was applied. The cracking algorithms were seen to be directly available for fibre-reinforced brittle materials, provided multi-linear constitutive behaviour is allowed as an input. The discrete crack analysis usually gives better results than the smeared crack model and the multi-surface plasticity model exhibits better convergence than the fixed smeared crack model.

11. The correct size effect of fibre-reinforced beams could be reproduced using the discrete crack concept.

12. The automatic generation routine greatly simplifies the creation of discrete crack elements. The same routine can be also used in the generation of reinforcing bar elements and other geometrical details.

13. Subjects left for further study include more accurate formulations for the effects of matrix tensile softening, debonding energy, and unloading on the fibre analysis.

14. In this study the applications were limited to cement-based composites. The model, however, is not restricted to these but may be used with other brittle matrix composites. The validity for analysing other composites, such as fibre-reinforced ceramics, has yet to be studied.

7. Summary of the appended papers

Paper 1 (App. I) deals with the fibre pullout problem in the full bond stage. Because perfect bond is assumed, the problem can be considered linear elastic. However, no elastic solution of the pullout problem is available. Therefore the finite element method is used to derive the relationship between the force and relative displacement at the fibre loaded end. This relation is then used in the one-dimensional fibre pullout theory to solve the bond modulus κ, which is defined as a slope between the interfacial shear stress and the fibre relative displacement. Its dependence on the fibre diameter, fibre volume fraction, and moduli of elasticity of the fibre and the matrix was studied. By using dimensional analysis a normalised function for the bond modulus could be evaluated, with two non-dimensional material parameters. A parametric study was then performed by varying the non-dimensional parameters in the finite element analysis and solving the bond modulus thereafter from the non-linear equation. A parametric plot was presented, from which a numerical estimation for the bond modulus can be read.

Paper 2 (App. II) introduces a fictitious two-fibre system for multiple fracture, where fibres can bridge several cracks. A full bond is assumed. In the two-fibre system two fibres are assumed to locate as mirror images of each other so that the force equilibrium at cracks is satisfied. A simplified one-dimensional model is chosen. The problem leads to a fourth order differential equation for the fibre force within the matrix segment between cracks. Equation can be solved, and the unknown coefficients determined, if the fibre forces at the cracks are known. The relations are then derived between the fibre forces and displacements at the cracks. The solutions from the two-fibre system are compared with those of the single fibre theory. Also studied is a case where the fibre embedded length is shorter than the distance between cracks.

Paper 3 (App. III) extends the fictitious two-fibre theory of Paper 2 by considering progressive debonding and frictional sliding. The bond between the two fibres and the matrix in a cross-section may be either elastic or frictional. It may also be a mixture, so that for one fibre the bond is elastic while for the other it is frictional. Corresponding equations are deduced and all possible bond combinations within a matrix segment evaluated. Fibre ends can introduce inaccuracy due to a simplified theory, in which the non-overlapping fibre segments are analysed independently as a single fibre.

When the relations between the fibre forces and displacements at the cracks are evaluated separately for every matrix segment, the fibre forces at the cracks can be solved using the compatibility condition of equal crack widths. Once the fibre forces at the cracks are solved, the stress distributions in the fibre, matrix and the interface can be evaluated. The problem is non-linear due to debonding, which occurs if the interfacial shear stress exceeds the bond strength.

Paper 4 (App. IV) describes how the statistical macromechanical behaviour can be deduced from the single fibre analysis. The computational model is fairly general, including both single and multiple fracture, the fracture mode being either fibre rupture or pullout. Moreover, hybrid composites with several fibre types can be analysed. Fibre content, length, or distribution are not restricted. On the other hand, fibres are assumed to be separate, straight, smooth, and fully flexible.

The fibre orientation introduces additional phenomena: number of fibres at the crack, snubbing effect, and matrix spalling. Their effects are modelled for every fibre orientation. By integrating over all fibre positions and averaging, the statistical stress in the composite can be evaluated. The cracking criterion is tested using an average tensile stress in the matrix halfway between the cracks. The result is a non-linear tensile stress–strain or stress–crack width relation of the composite.

Paper 5 (App. V) is an application study in which the statistical tensile behaviour is extended into two dimensions using different cracking models in the DIANA finite element package. The tensile model represents the stress–crack opening relation. In case of a strain hardening composite with multiple fracture two stages must be distinguished. In the hardening stage the strain is a material property, whereas in the softening state the strain depends also on the element size. The three approaches used in this study are the smeared and discrete crack concepts and a multi-surface plasticity model.

It has been suggested that a smeared crack predictor analysis be done first, followed by a discrete crack corrector analysis. Because modelling of the interface elements representing discrete cracks leads to re-modelling of the geometry, a method is introduced with which the interface elements can be generated on the existing geometry. Moreover, based on the same algorithm the reinforcing bars can be created with interface elements connecting them to the plane elements and representing the bond–slip behaviour.

The applications are flexural members subjected to proportional loading: a steel fibre-reinforced beam which exhibits multiple fracture, kevlar fibre beams with and without ordinary reinforcement, and a concrete pipe under a diametral line load, either without fibres or reinforced with Wirex or Dramix fibres. The tensile behaviour of all materials was computed from the micromechanical properties using the proposed model. The behaviour observed in the experiments could be reproduced, which supports the validity of these methods for analysing fibre-reinforced brittle materials.

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